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Control Uncertainty in Fine Motion Planning

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CONTROL UNCERTAINTY
IN
FINE MOTION PLANNING

A DISSERTATION
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Abstract

The goal of *Fine Motion Planning* is to generate provable robot programs in the presence of control, sensing, and model uncertainty. In particular, fine motion planning includes part mating sequences that require force control. We first present a new result in the formalism of fine motion planning that, embedding the knowledge of the termination condition in the construction of preimage, augments the size of preimage. We, then, present new results on the characterization of control uncertainty.

Define *forward projection* of a state to be the union of all possible states such that there exists a trajectory connecting the two states for controls that remain within a given model of control uncertainty. Computing forward projection is a problem in differential inclusion. Previous work in this area by Blagodat·skikh and Filippov (1986) allows one to compute the boundary of *attainable set* that are states reachable at *a given instant.*

We consider computing the *boundary* of forward and backprojection on smooth sets of an arbitrary but autonomous control scheme. The control equations include state-dependent differential equations governing general rigid body motion, in contact with other rigid bodies or moving freely. We characterize the boundary as an *integral manifold* of a Hamilton-Jacobi equation (Butkovskii, 1982). This integral manifold is a solution of a system of $2n$ ordinary differential equations where n is the dimension of the state space. We give conditions for the *existence* and *uniqueness* of the *local boundary* of the forward and backprojection of a state described by a *regular closed subset.* The *Global boundary* may have several connected components. Some of these are subsets of the boundary of perturbations of the *unstable* and the *stable manifold* of saddle type singularity. We also give conditions for the *existence* and *uniqueness* of the local boundary of the perturbations of the *unstable* and the *stable manifold.* Thus, when *invariant* subsets of the differential inclusion problem are trivial, our conditions imply *locally finite representation* and *computability* of control uncertainty.

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Chapter 1

Introduction

Fine Motion Planning, also known as *Motion Planning with Uncertainty*, considers a system whose state evolves according to some known physical law and is equipped with sensors that allows it to ascertain its state at any instant. Neither parameters describing the evolution of the state are known exactly, nor do sensors determine the state precisely. However, it is conceivable that these errors follow some probabilistic distribution. It is also possible to add an additional constraint that the environment in which the system evolves can only be described approximately, perhaps given by some known distribution. We consider then the following geometric planning problem:

> *The system starts somewhere, it can ascertain its starting state using its sensors; it needs to get to some goal; it must synthesize a control law or a sequence of control laws to reach the goal and some evaluation function or a sequence of evaluation functions on the set of sensing queries that can be interpreted as having attained the goal or intermediate goals without knowing the environment, the evolution of control trajectory, or sensing queries in their respective distributions a-priori.*

There are several variants of the fine motion planning problem. First, let us assume that the errors in evolution parameters, sensing, and the environment are described by *uniform density distributions* - equal probability over some closed and bounded domain. If for all possible evolutions, sensing queries, and the environment within

their bounds the system is guaranteed to reach and terminate in the goal, it is called *Strongly Guaranteed Plan.* If there exists at least one evolution of the system, a set of sensing queries, and the environment within prescribed uniform density distributions such that the system reaches and terminates in the goal, the plan is a *Weakly Guaranteed Plan.* A variant of a weakly guaranteed plan is also called Error Detection and Recovery. Finally, consider that the errors are instead described by an arbitrary probabilistic distribution. The success of reaching and terminating in the goal can possibly be probabilistically characterized and such plans would be *Probabilistic Plans.* In this general probabilistic sense, strongly guaranteed plans are those that have one hundred percent probability of success and weakly guaranteed plans are those that have some non-zero probability of success when errors are known to belong to some uniform density distribution. We consider strongly guaranteed plans in a perfectly well known environment - one hundred percent probability of success when uncertainty in sensing and control belong to uniform density distribution.

1.1 Results

The domain of Strongly Guaranteed Fine Motion Planning with uncertainty in sensing and control is described by the following:

- `Shape of the Moving Objects and Obstacles`,
- `Configuration Space`,
- `Control Scheme and its Model of Uncertainty`, and
- `Model of Sensing and its Uncertainty`.

The set of independent parameters that uniquely determine the position and orientation of all moving objects define the *configuration space.* The shape of the moving objects and obstacles determine a subset of the configuration space called *valid configuration space* where the rigid body extent of the moving objects do not violate the rigid body extent of the obstacles. The valid configuration space consists of the free space where the moving body does not touch any of the fixed obstacles and the

boundaries of the configuration space obstacles where the moving body touches one or several of the fixed obstacles.

A nominal behavior of a feedback control system either in the free space or on the boundaries of the configuration space obstacles is described by a system of differential equations. We consider *autonomous* differential equations: the single valued map that assigns to each state a velocity (acceleration) depends only on the state of the system but not on time. A possible perturbations of a nominal feedback control system is described by a multivalued map that assigns to each state a subset of the velocities (accelerations).

The uncertainty in sensing is described similarly by a multivalued map that assigns to every point in the domain a subset of the range: the subset of possible sensing values that a sensor at a configuration is likely to return.

The main contributions of the thesis are presented in two subsections following a subsection on the notations and definitions.

1.1.1 Preliminaries

Let M denote a smooth subset of the valid configuration space. Examples of such subsets include spaces where a moving object touches obstacles. If we consider second-order differential equations, then for notational simplicity we use M itself to denote TM, the tangent bundle of M. Let the domain for fine motion planning be a compact subset V of M.

A control statement **CS** determines uncertainty in a control scheme defined by $F^0: V \to TM$, a multivalued section of the tangent bundle. Let all absolutely continuous solutions, $\phi: \mathbf{R} \times V \to M$, to the corresponding Cauchy problem (see equation 2.26) at $\mathbf{s} \in V$ be a set denoted $\Phi(\mathbf{s})$. Let $\mathcal{J}_{\phi\mathbf{s}}$ denote the interval of time for which a solution $\phi_\mathbf{s}$ is defined over V. The *orbit* of a solution ϕ at a point $\mathbf{s}$ is the set $\{\phi(t, \mathbf{s}) | t \in \mathcal{J}_{\phi\mathbf{s}}\}$.

Let the model of uncertainty in sensing be also defined by a multivalued map $\mathcal{K}^*: M \to \mathbf{E}$, where $\mathbf{E}$ is the space of measurements. Let us denote a measurement by $\mathbf{m}^*$ so that $\mathbf{m}^*(\mathbf{s}) \in \mathcal{K}^*(\mathbf{s})$. Let $\mathbf{m}^*_{\phi_\mathbf{s}}$ denote sensing measurements along a trajectory $\phi_\mathbf{s}$, *i.e.*, $\mathbf{m}^*_{\phi_\mathbf{s}}: \mathbf{R} \to \mathbf{E}: t \mapsto \mathbf{m}^*(\phi(t, \mathbf{s})) \in \mathcal{K}^*(\phi(t, \mathbf{s}))$. Note that $\mathbf{m}^*_{\phi_\mathbf{s}}$ need not be a

continuous function of time. A history of sensing measurement from time 0 to t is defined as $\mathbf{m}^*_{\phi s}[0,t] = \{\mathbf{m}^*(\phi(t',\mathrm{s}))|t' \in [0,t]\}$.

A termination condition **TC** determines a termination predicate **tp**, a boolean valued function, defined on the set of subsets of the space of measurements, *i.e.*, $\mathbf{tp}: 2^{\mathbf{E}} \rightarrow \{true, false\}$.

A motion command **M** is a tuple (**CS**, **TC**) consisting of a control statement **CS** and a termination condition **TC**.

1.1.2 One-Step Preimage

Let $\mathcal{I}$ and $\mathcal{G} = \{\mathcal{G}_\alpha\}$ be subsets of V - a set of initial configurations and a set of goal configurations, respectively. Then, *Fine Motion Planning* inputs are

$$\{M, V, F^0, \mathcal{K}^*, \mathcal{I}, \mathcal{G}\}. \tag{1.1}$$

Define the *Backprojection* of a goal $\mathcal{G}$ for a control statement **CS** to be $\{\mathrm{s}|\forall\phi_{\mathrm{s}} \in \Phi(\mathrm{s}), \exists t \geq 0, \phi(t,\mathrm{s}) \in \mathcal{G}\}$. They are states such that all possible evolutions following the control statement **CS** reach the goal at some time. Define a predicate $\texttt{Achieve}(\mathcal{G},\mathbf{M},\mathcal{I})$ encoding the condition that a motion command **M** is guaranteed not only to reach $\mathcal{G}$ but also terminate in $\mathcal{G}$ whenever the starting configuration is in $\mathcal{I}$:

$$\begin{aligned}
\texttt{Achieve}(\mathcal{G},\mathbf{M},\mathcal{I}) = \{\mathrm{s} \in \mathcal{I}|\forall\phi_{\mathrm{s}} \in \Phi(\mathrm{s}),& \\
&(i)\quad Let\quad \Omega = \{t \in \mathcal{J}_{\phi\mathrm{s}}|\forall \mathbf{m}^*_{\phi\mathrm{s}}[0,t], \mathbf{tp}(\mathbf{m}^*_{\phi\mathrm{s}}[0,t]) = true\} \\
&\qquad t_f = \begin{cases} sup(\mathcal{J}_{\phi\mathrm{s}}), & \text{if } \Omega = \emptyset; \\ inf(\Omega), & \text{otherwise.} \end{cases} \\
&\qquad \lim_{t\to t_f} \phi_{\mathrm{s}}(t) \in \mathcal{G}, \\
&(ii)\quad 0 \leq t < t_f,\ \exists \mathbf{m}^*_{\phi\mathrm{s}}[0,t],\ \mathbf{tp}(\mathbf{m}^*_{\phi\mathrm{s}}[0,t]) = true \\
&\qquad \Rightarrow \phi_{\mathrm{s}}(t) \in \mathcal{G}\}
\end{aligned} \tag{1.2}$$

A One-Step Preimage $\mathcal{P}_{\mathbf{M}}(\mathcal{G})$ of a goal $\mathcal{G} = \{\mathcal{G}_\alpha\}$ are those points that satisfy $\texttt{Achieve}(\mathcal{G},\mathbf{M},\mathcal{P}_{\mathbf{M}}(\mathcal{G}))$. A maximal preimage $\mathcal{P}^{MAX}_{\mathbf{M}}(\mathcal{G})$ are all those points s that satisfy $\texttt{Achieve}(\mathcal{G},\mathbf{M},\{\mathrm{s}\})$. A simple illustration of a preimage is given by example 2.1(a). We present the following results on One-Step Fine Motion Plans:

- A precise construction of a preimage accounting for possible initial states, and

- An observation that embedding the knowledge of a termination predicate in the construction of a preimage augments the size of the preimage

(see section 2.4). A definition of preimage was initially given by Lozano-Pérez, Mason, and Taylor [LMT 84]. Though our definition, developed along the lines of a definition given by Latombe [Lat 88, Lat 91], also includes classical control theoretic fine motion plans of the kind considered in example 1.1 and 2.1(c). That the set of initial states determines possible interpretations of sensory measurements is an observation given by Lozano-Pérez, Mason, and Taylor [LMT 84], but the fact that the termination condition also affects possible interpretations is a new observation. As a result, a significantly larger preimage can be constructed. An illustrative example is given in Section 2.4. Some of these results and examples have appeared earlier [LLS 89, LLS 91, Lat 91, SL 91].

1.1.3 Characterization of Control Uncertainty

A set of perturbations of a control equation determines a differential inclusion problem given by the map F^0. A differential inclusion problem is a direct generalization of ordinary differential equations - instead of just a curve whose tangent is given, a set of curves are solutions whose tangents lie in the image of a multivaled map. We consider solutions that are absolutely continuous functions: the solution is continuous and the set of discontinuities of the derivative of the solution is a set of measure zero. Consider those states reachable by a possible evolution. They are called a *Reachable Set.* Two kinds of reachable sets can be defined: an *Attainable Set* contains states reachable at any given instant, and a *Forward Projection Set* is the union of states reachable at all future times including the present. We consider determining the boundary of such sets: $\delta\mathcal{A}(t,\mathcal{I})$, the boundary of attainable set at time t for initial states in $\mathcal{I}$, and $\delta FP(\mathcal{I})$, the boundary of the forward projection of initial states in $\mathcal{I}$. As a consequence of characterizing $\delta FP(\mathcal{I})$, the boundary of the forward projection, we characterize the boundary of the *backprojection* of a goal $\mathcal{G}$.

When multivalued map F^0 is *non-empty, closed, Lipschitzian*, and *autonomous*, the set of absolutely continuous solutions $\Phi(\mathbf{s})$ are path connected [StWu 91]. Let H be a smooth Hamiltonian function defined on the cotangent bundle T^*M of M. A subset of the cotangent bundle T^*M where the Hamiltonian H remains constant is called a *characteristic* of H and is denoted by $\mathcal{C}$. Any subset of M whose covariant section is a subset of the characteristic set $\mathcal{C}$ is called an *Integral Manifold.* It is a property of the Hamiltonian vector field X_H that their time evolutions preserve the integral manifolds and the characteristic set $\mathcal{C}$ of H.

Let H be a Hamiltonian defined by $H: T^*M \to \mathbf{R}: \mathbf{w}_\mathbf{s}^* \overset{H}{\mapsto} sup_{\mathbf{v_s} \in F^0(\mathbf{s})} w_\mathbf{s}^*(\mathbf{v_s})$. Two results given by Blagodat·skikh and Filippov [BlFi 86], and Butkovskii [But 82] show that the solution set of the Hamiltonian vector field X_H of H describe the boundary of the attainable set and the boundary of the forward projection, respectively, when the initial state $\mathcal{I}$ is a state $\{\mathbf{s}\}$.

Our first result, a theorem characterizing the initial set that locally defines the boundary of the forward projection and the boundary of the backprojection of a *regular closed* set $\mathcal{I}$, is a direct application of the Hamilton-Jacobi theorem to Hamiltonian systems:

Theorem 1.1 *Let zero be a regular value of a smooth map $f_\mathcal{I}: M \to \mathbf{R}$ so that $\delta\mathcal{I} = f_\mathcal{I}^{-1}(0)$ is the boundary of the set $\mathcal{I}$. Let $df_\mathcal{I}: M \to T^*M$ be the corresponding covariant section. Let zero be a regular value of the smooth map $f_\mathcal{I} \times H \circ df_\mathcal{I}: M \to \mathbf{R}^2: \mathbf{s} \mapsto (f_\mathcal{I}(\mathbf{s}), H \circ df_\mathcal{I}(\mathbf{s}))$. Then, $\Lambda = (f_\mathcal{I} \times H \circ df_\mathcal{I})^{-1}(0)$ is a well-defined $(n-2)$-dimensional submanifold and a subset of the characteristic set $\mathcal{C}$.*

The integral manifold of Hamiltonian Vector Field X_H with initial set Λ_i, an appropriate subset of Λ, locally defines the boundary of the forward projection (see section 3.2.6). The local boundary of the backprojection is similarly defined with the negative set valued map $-F^0$ and a subset Λ_i of the corresponding Λ (see Figure 3.2 and Example 3.2 for the initial manifold Λ of a backprojection boundary).

Consider a single valued continuous time dynamical system [PalMel 82]. The *ω-Limit Set* of a point $\mathbf{u}$ are those points $\mathbf{s} \in M$ for which there exists a sequence $t_i \to \infty$ such that $\phi(t_i, \mathbf{u}) \to \mathbf{s}$. The *$\alpha$-Limit Set* is defined analogously for $t_i \to -\infty$.

It is apparent that the *ω-Limit Set* and *α-Limit Set* can be defined for orbits of a point and the *ω-Limit Set* of a point for a field X is *α-Limit Set* for the field $-X$. If s is a singular point of a field X, then the set of points of M which have s as *ω-Limit Set* is called the *Stable Manifold* denoted $W^s(\mathbf{s})$. The unstable manifold $W^u(\mathbf{s})$ is defined similarly as those points in M whose *α-Limit Set* is s. The *Stable Manifold Theorem* [PalMel 82] establishes that they are C^r injectively immersed manifolds if the vector field is C^r. The stable manifold of a *sink* singular point is also called the *basin of attraction* of the sink. The local stable (unstable) manifolds $W_r^s(\mathbf{s})$ ($W_r^u(\mathbf{s})$) for a ball $B_r(\mathbf{s})$ centered at s of radius $r > 0$ are those points in the stable (unstable) manifolds whose positive (negative) orbits remain in the ball $B_r(\mathbf{s})$. The local stable and unstable manifolds are topologically embedded discs.

Consider now the differential inclusion problem $\dot{\mathbf{s}} \in F^0(\mathbf{s})$. If the orbit of all possible absolutely continuous functions that are solutions to the differential inclusion problem remain in a subset, then the subset is a *strongly invariant* subset. If there exists at least a solution such that its orbit remains in the subset, then it is a *weakly invariant* subset. Consider a point $\mathbf{s} \in V$ such that $0 \in F^0(\mathbf{s})$. A possible flow is the trivial one - a constant map, $\phi(t, \mathbf{s}) = \mathbf{s}$, defined for all times. Such points are, therefore, at least weakly invariant. Let states $\mathbf{s} \in V$ such that $0 \in F^0(\mathbf{s})$ be called *Singular Invariant Subset.* We denote the union of all such singular invariant subsets by $\mathcal{Z}$. Section 3.2.3 considers conditions when such subsets can be uniformly labeled as of type *sink*, *source*, or *saddle*.

Now, consider computing the forward projection of a singular invariant set. This boundary of the forward projection is also the boundary of the perturbation of unstable manifolds. The perturbation of *stable manifold* is the forward projection of the negative field $\dot{\mathbf{s}} \in -F^0(\mathbf{s})$. If we consider the differential inclusion problem as an instance of a set of perturbations of an arbitrary autonomous dynamical system, then the basin of attraction of a *sink singular invariant subset* is a natural candidate for the preimage. If the sink singular invariant subset is not strongly invariant, some strongly invariant subset in the basin of attraction containing the sink singular invariant subset is a goal of the basin of attraction with a perpetually false termination predicate. Some components of the boundary of such basins of attraction are the boundary of

perturbations of the stable manifold. In addition, also consider the forward and the back projection of a regular closed set. The global boundary may have components that is the boundary of perturbations of the unstable manifold of the singularities of the type saddle. Such additional components are present in the description of the forward projection of points that lie in the perturbation of the *stable manifolds* of the saddle. Such components of the boundary of the forward and the back projection are distinct from the local component given by the Theorem 1.1. We consider computing the local boundary of the perturbation of stable and unstable manifolds of singular invariant subsets, particularly those of the type saddle.

We show that when the boundary, δF^0, of a regular, closed, non-empty, multi-valued map F^0 is smooth, the singularities of the corresponding Hamiltonian vector field X_H defined above are degenerate. Consider instead, *regular, convex polytopes* as the boundary of the image of the map F^0 at all points. Then, there exists Hamiltonian vector fields X_H that are non-degenerate at singularities. With additional conditions given in the Proposition 1.1 below, the boundary of the forward projection of a singular invariant set is characterized by Theorem 1.2.

Let the faces of the convex polytope in the image of the map F^0 be denoted $\mathcal{P}_{i_1 \ldots i_m}$. The linear span of all vector fields in a face determines a distribution. We assume that each such distribution is *integrable* in the sense of Frobenius [Spiv 79]. Let $\mathcal{Z}_j$ denote the j^{th} singular invariant set. Let $\mathcal{T}^j_{i_1 \ldots i_m}$ denote a face of the boundary $\delta \mathcal{Z}_j$ of the singular invariant set corresponding to the face $\mathcal{P}_{i_1 \ldots i_m}$ (see figure 1.1). Consider a point $\mathbf{s}$ on the face $\mathcal{T}^j_{i_1 \ldots i_m}$. Let $A = DX(\mathbf{s})$ be the linearization of a field X such that $X \in \mathcal{P}_{i_1 \ldots i_m}$ in some local neighborhood, $X(\mathbf{s}) = 0$, and $\mathbf{s}$ is a hyperbolic singularity of X. The linear mapping A has a natural splitting into A_u and A_s, the expansion and contraction part respectively, such that $T_{\mathbf{s}}M = E^u \oplus E^s$, $A_u = A|_{E^u}$, and $A_s = A|_{E^s}$. Each of these subspaces E^u and E^s are invariant.

Proposition 1.1 *Let point* $\mathbf{s}$ *lying on* $\mathcal{T}^j_{i_1 \ldots i_m}$ *be a hyperbolic singularity of* X. *If*

1. *The unstable subspace* E^u *is a supporting subspace to* $\mathcal{Z}_j$ *at* $\mathbf{s}$,

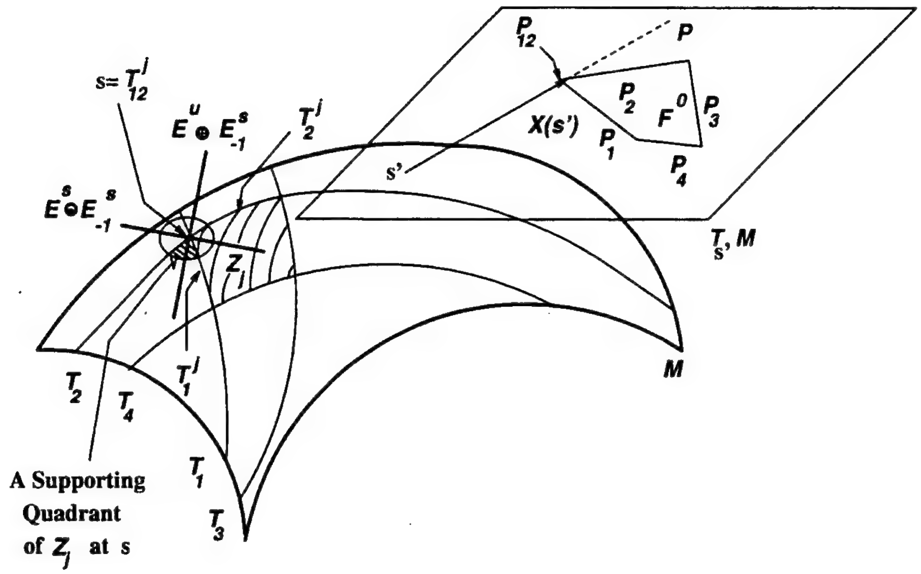


Figure 1.1: Polytope Neighborhood of a Vector Field and Transition Hypersurfaces

2. *A codimension one* invariant subspace *denoted E^s_{-1} of stable subspace E^s is contained in the tangent space $TT^j_{i_1 \ldots i_m}$, and the remaining one-dimensional subspace, $E^s \ominus E^s_{-1}$, of the stable subspace is not contained in the tangent space $TT^j_{i_1 \ldots i_m}$,*

then, the point s *belongs to the zero set of the characteristic $\mathcal{C}$ of the function H, i.e., $H(\mathrm{s}, \alpha) = 0$, where* $\ker(\alpha) = E^u \oplus E^s_{-1}$, $\alpha \neq 0$.

Section 3.2.10 contains more details on further splitting of each of the subspaces E^s and E^u into invariant subspaces and its implications on the structure of the codimension one invariant subspace E^s_{-1}.

Corollary 1.1 *If the points on two segments $T^j_{i_1 \ldots i_m}$ and $T^j_{k_1 \ldots k_q}$ lie in the zero set of the characteristic $\mathcal{C}$ of the function H as in Proposition 1.1, then points on a segment in their intersection also lie in the zero set of the characteristic $\mathcal{C}$.*

Now, consider defining locally the boundary of the forward projection of a singular invariant set.

Theorem 1.2 *Let Λ_j be a subset of $\delta \mathcal{Z}_j$ so that the conditions of the Proposition 1.1, or the Corollary 1.1 are satisfied on Λ_j. In addition, assume that $E^u(\mathrm{s})$ is a* strictly

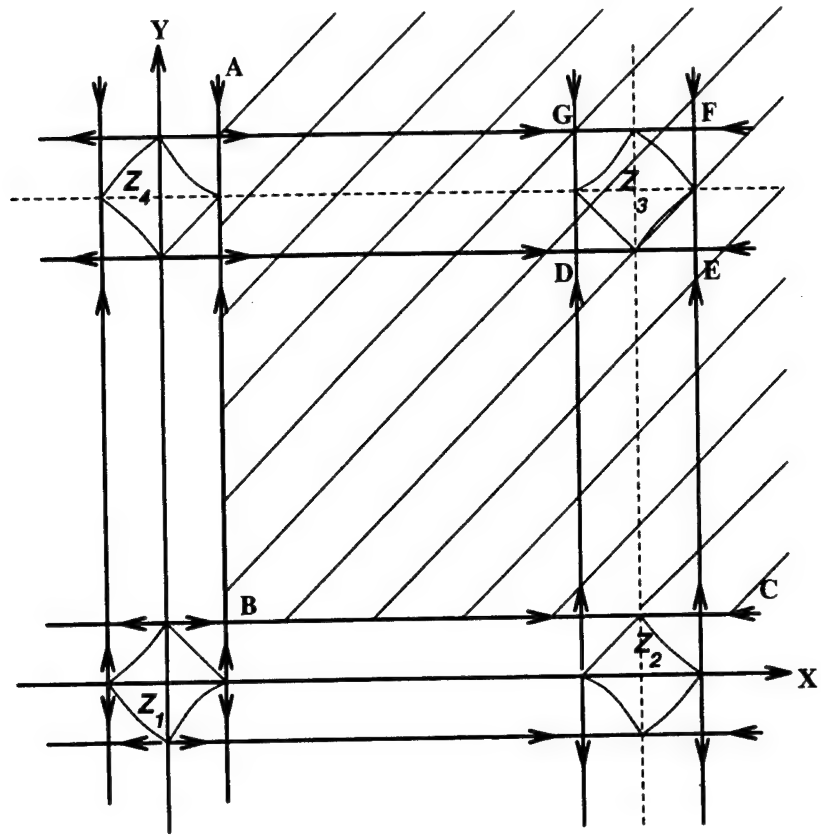


Figure 1.2: Hatched region is Preimage of Rectangle $DEFG$ for **TC**={false}

supporting subspace *on* Λ_j. *Then, for some* $r > 0$, *the local unstable manifold* $W_r^u(\Lambda_j)$ *is an* integral manifold *and defines the boundary of the forward projection of* $\mathcal{Z}_j$.

The notion of strictly supporting subspace is similar to that of a supporting plane to a convex polytope. We give a precise definition in the Section 3.2.9.

Example 1.1 Consider the Euclidean plane $M = \mathbf{R}^2$. Consider a neighborhood F^0 defined by the intersection of the positive half spaces of the following four hyperplanes:

$$\begin{aligned}
\mathcal{P}_1 &= \{(-(x^2-x)-\epsilon_x)\frac{\partial}{\partial x} - (y^2-y)\frac{\partial}{\partial y}, \epsilon_y dx - \epsilon_x dy\}, \\
\mathcal{P}_2 &= \{-(x^2-x)\frac{\partial}{\partial x} + (-(y^2-y)+\epsilon_y)\frac{\partial}{\partial y}, -\epsilon_y dx - \epsilon_x dy\}, \\
\mathcal{P}_3 &= \{(-(x^2-x)+\epsilon_x)\frac{\partial}{\partial x} - (y^2-y)\frac{\partial}{\partial y}, -\epsilon_y dx + \epsilon_x dy\}, \qquad (1.3)
\end{aligned}$$

$$\mathcal{P}_4 = \{-(x^2-x)\frac{\partial}{\partial x} + (-(y^2-y)-\epsilon_y)\frac{\partial}{\partial y}, \epsilon_y dx + \epsilon_x dy\},$$

where $\epsilon_x, \epsilon_y > 0$ are constants. This is a neighborhood of a nominal vector field $-(x^2-x)\frac{\partial}{\partial x} - (y^2-y)\frac{\partial}{\partial y}$. The singular invariant sets are regions $\mathcal{Z}_i, i = 1,4$, centered at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, respectively. Region $\mathcal{Z}_1$ is of type *source*, $\mathcal{Z}_3$ is of type *sink*, and $\mathcal{Z}_2, \mathcal{Z}_4$ are of type *saddle*. The boundary of the backprojection of the sink $\mathcal{Z}_3$ is locally null and it is possible to verify that $\mathcal{Z}_3$ is not strongly invariant subset.

However, it can also be verified that the rectangle $DEFG$ containing $\mathcal{Z}_3$ in Figure 1.2 is a strongly invariant subset (see conditions of Aubin and Cellina [AubCel 84] and Section 3.2.6). If **TC**={false} is the termination condition, then the open infinite hatched subset of the plane is a basin of attraction of the strongly invariant subset $DEFG$, and therefore, a preimage. Notice that the boundary of the basin of attraction is a piecewise smooth segment labeled ABC which is a component of the boundary of perturbations of the stable manifold of the saddle singularities $\mathcal{Z}_2$ and $\mathcal{Z}_4$ (see example 3.4 for more details).

1.2 Motivation

An autonomous robot must plan its actions from a sufficiently high-level instructions. The high-level instructions require geometrical tools to translate them to a manipulator-level program. Significant results in the problem of motion planning to avoid obstacles have been achieved in the last decade [Lat 91]. However, sensors are never perfect and if a controller could be designed with perfect tracking ability and the models could be manufactured/measured perfectly and placed in the environment with preciseness, motion planning as given above would suffice. In fact, if the relative magnitude of the errors remain small in relation to the closeness of the representative path with the obstacles, such programs would still work. Examples are the commonly available painting and welding robots that track representative paths that are offset from the fixed objects a distance reflecting the aggregate expected error. Another example is a peg assembly by RobotWorld^{TM} using RISC (Reduced Intricacy

in Sensing and Control) [CanGol] that reduces execution errors with just-in-time sensing to within $1/1000^{th}$ of an inch - better than the aggregate sum of tolerance of parts, fixturing, and errors in RobotWorld^{TM}. However, in many applications the preciseness with which a pre-planned path can be executed in relation to the model of the environment and the specification of the goal is several orders of magnitude worse. Such applications include contact type tasks such as dextrous manipulation, precision assembly requiring force control, and mobile robot navigation. The level of resolution at which such applications are executed also entails the impreciseness in sensing (*i.e.,* localization), geometrical tolerances of the shape and size of the environment, the mechanics of motion in the presence of friction, and the transition from one contact type to another. In presence of all these uncertainties, the planner should determine if an objective given by the high level instructions can be accomplished and determine such a plan.

1.2.1 A Motivational Example

Some typical examples are those of a mobile robot or a planar peg-in-hole assembly. We consider a situation in the planar peg-in-hole problem when the equations of motion are described on a constrained surface - in the absence of any constraint, our example reduces to the motion of a peg in a free floating space.

The peg is a moving body described as a polygon and the hole is an obstacle also described as a polygon as shown in Figure 1.3(a). The corresponding configuration space obstacle shown in Figure 1.3(b) is a subset of a three-dimensional Euclidean space for the two-degrees-of-freedom of translation and one-degree-of-freedom of rotation of the peg in the plane. The configuration space obstacle is described by several sets of equations each describing a smooth subset of the obstacle. All together they describe a *valid* subset of the configuration space - the region where the peg's rigid body extent does not violate the rigid body extent of the obstacles. In effect, the region where the peg's spatial extent intersects with the spatial extent of the obstacles is removed from the total space to obtain a description of the valid configuration space. The valid configuration space is made up of subsets of smooth

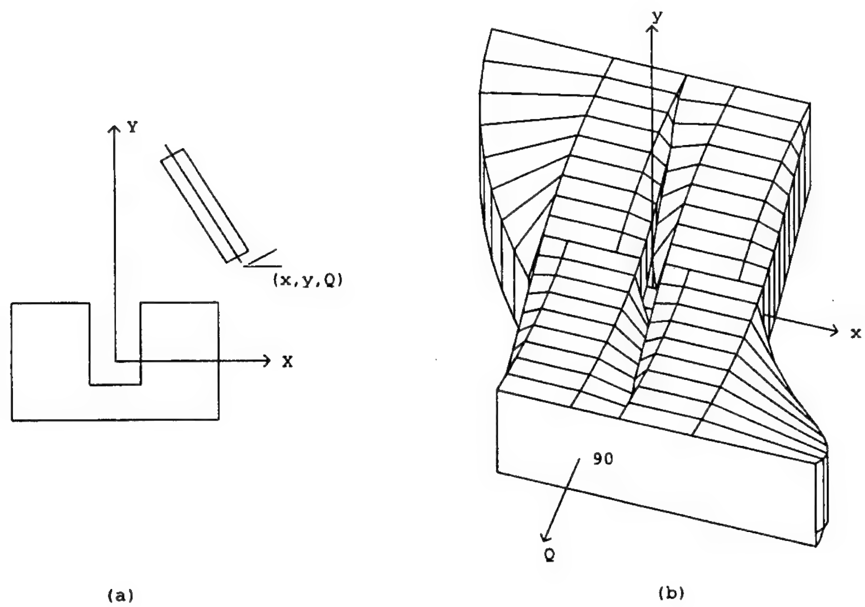


Figure 1.3: Configuration Space of a Typical Peg-in-Hole Assembly

configuration space constraint surfaces, each such smooth surface is defined by zero or finitely many equations. Let us denote such smooth configuration space constraint surfaces by M. Consider a specific example of a configuration space constraint surface, of type A in the notation of Lozano-Pérez [Loz 83], where an edge of the moving body is in contact with a vertex of an obstacle. The configuration of the center of mass of the moving body denoted $\mathbf{q} = (x, y, \theta)$, in type-A contact is constrainted to $g(\mathbf{q}) = x\sin(\theta) - y\cos(\theta) + a = 0$ in an appropriate choice of coordinate system (see section 2.1.2). For the rest of the thesis, we consider one such smooth subset M as our domain.

Let a configuration constraint surface M be given by

$$g(\mathbf{q}) = 0 \tag{1.4}$$

where $\mathbf{q}$, a subset of $\mathbf{R}^m$, is a generalized configuration vector and $g\colon \mathbf{R}^m \to \mathbf{R}^l$ is a function with zero as its regular value. Equation 1.4 describes a $(m-l)$-dimensional

constraint space. The velocity and acceleration constraints are given by time derivatives of equation 1.4 and look as follows when arranged appropriately:

$$J(\mathbf{q})\dot{\mathbf{q}} = 0 \tag{1.5}$$

$$J(\mathbf{q})\ddot{\mathbf{q}} + h(\mathbf{q}, \dot{\mathbf{q}}) = 0 \tag{1.6}$$

For a constraint of type-A, given earlier, the velocity and acceleration constraints are

$$[\sin(\theta) \quad -\cos(\theta) \quad (x\cos(\theta) + y\sin(\theta))][\dot{x} \quad \dot{y} \quad \dot{\theta}]^T = 0,$$

and

$$\begin{aligned} [& \sin(\theta) \quad -\cos(\theta) \quad (x\cos(\theta) + y\sin(\theta))][\ddot{x} \quad \ddot{y} \quad \ddot{\theta}]^T \\ + & (\dot{\theta}^2(-x\sin(\theta) + y\cos(\theta)) + 2\dot{\theta}(\dot{x}\cos(\theta) + \dot{y}\sin(\theta))) = 0. \end{aligned}$$

In the absence of friction at the contact, the equations of motion of the body moving with constraint 1.4 is given by

$$\Lambda(\mathbf{q})\ddot{\mathbf{q}} + \mu(\mathbf{q}, \dot{\mathbf{q}}) + p(\mathbf{q}) = \mathbf{f} + J^T(\mathbf{q})\lambda, \tag{1.7}$$

where $\Lambda(\mathbf{q})$ is an $m \times m$ symmetric inertia matrix, $\mu(\mathbf{q}, \dot{\mathbf{q}})$ is the vector of centrifugal and Coriolis forces, $p(\mathbf{q})$ is gravitational force term, $\mathbf{f}$ is a vector of externally applied generalized forces, $J^T(\mathbf{q})$ is the transpose of the Jacobian matrix defined in equation 1.5 and $\lambda \in \mathbf{R}^l$ is a vector of Lagrangian multipliers. The m scalar equations in 1.7 and l equations in 1.6 form a square linear system with $m + l$ unknowns $(\ddot{\mathbf{q}}, \lambda)$. This system can be solved for the unknowns in terms of quantities $\mathbf{s} = (\mathbf{q}, \dot{\mathbf{q}})$, called the state. This defines a vector field

$$\dot{\mathbf{s}} = X(\mathbf{s})$$

on a manifold TM defined by equations 1.4 and 1.5, *i.e.*, $TM \equiv \{\mathbf{s} = (\mathbf{q}, \dot{\mathbf{q}}) | g(\mathbf{q}) = 0, J(\mathbf{q})\dot{\mathbf{q}} = 0\}$. The Lagrangian multipliers are to be interpreted as normal forces. If the configuration constraint is a passive constraint such as a free-flying peg under a type A constraint, then for a motion to preserve this constraint, the normal forces must point outwards from the surface. This condition translated in terms of the

Lagrangian multipliers defines a subset of the manifold TM where any actual motion occurs.

Let us consider a specific example of the equations of motion when the free-flying peg in the plane as considered earlier is moving on a frictionless constrained surface of type A with *force and moment control* of the type described by Khatib [Khat 87], Khatib and Burdick [KhBur 86], and Shekhar and Khatib [SK 87]. Let $\mathbf{q}_d$, $\dot{\mathbf{q}}_d$, $\ddot{\mathbf{q}}_d$, and $\mathbf{f}_d$ denote the vector of desired configuration, velocity, acceleration, and forces. Let the external force vector $\mathbf{f}$ applied at the center of mass of the moving peg be an orthogonal combination of components determined from a position control law and force control law. Let Ω and $\tilde{\Omega} = I - \Omega$ denote the selection matrices for position control and force control directions respectively. A particular example of Ω for the planar peg-in-hole example is a diagonal matrix $diag\{s_x, s_y, s_\theta\}$, where s_x, s_y, or $s_\theta = 1$ indicates that coordinate axis x, y, or θ is position controlled. The external force vector $\mathbf{f}$ for this example is

$$\begin{aligned}\mathbf{f} &= \Lambda(\mathbf{q})\Omega(\ddot{\mathbf{q}}_d + \mathbf{K}_p(\mathbf{q}_d - \mathbf{q}) + \mathbf{K}_v(\dot{\mathbf{q}}_d - \dot{\mathbf{q}})) + \tilde{\Omega}(-\mathbf{K}_{vf}\dot{\mathbf{q}} + \mathbf{f}_d + \mathbf{K}_f(\mathbf{f}_d - J^T(\mathbf{q})\lambda)) \\ &\stackrel{def}{=} \mathbf{f}^* - \tilde{\Omega}\mathbf{K}_f J^T(\mathbf{q})\lambda \qquad (1.8)\end{aligned}$$

where $\mathbf{f}^*$ is defined by equating the right-hand side of the first line with the second, and $\mathbf{K}_p$, $\mathbf{K}_v$, $\mathbf{K}_{vf}$, and $\mathbf{K}_f$ are position, velocity, force control velocity damping, and force gain matrices - of three-by-three in the planar peg-in-hole example. The term $J^T(\mathbf{q})\lambda$ represents the measured force of reaction coming from the derivative force feedback term. Combining equation 1.7 with $\mathbf{f}$ from equation 1.8 and equation 1.6 one obtains the desired linear system

$$\begin{bmatrix}\Lambda(\mathbf{q}) & (\tilde{\Omega}\mathbf{K}_f - I)J^T(\mathbf{q}) \\ J(\mathbf{q}) & 0\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}} \\ \lambda\end{bmatrix} = \begin{bmatrix}\mathbf{f}^* - \mu(\mathbf{q},\dot{\mathbf{q}}) - p(\mathbf{q}) \\ -h(\mathbf{q},\dot{\mathbf{q}})\end{bmatrix}. \qquad (1.9)$$

For the specific planar example of type A constraint we have been considering, the mass matrix $\Lambda(\mathbf{q})$ is a diagonal matrix $diag\{m, m, \hat{I}\}$ where $\hat{I}$ is the rotational inertia about the center of mass, $\mu(\mathbf{q},\dot{\mathbf{q}}) = 0$, $p(\mathbf{q})$ is defined according to if gravity has a component in the plane of motion, and $h(\mathbf{q},\dot{\mathbf{q}}) = \dot{\theta}^2(-x\sin(\theta) + y\cos(\theta)) + 2\dot{\theta}(\dot{x}\cos(\theta) + \dot{y}\sin(\theta))$.

For a given control law of equation 1.8, the solution of linear system 1.9 contains m second-order differential equations governing the motion of a body in type A contact. The solution of such differential equations describes evolutions of the system for all initial states $\mathbf{s}(0)$ in the domain. Let us denote ϕ as a solution. If the state of the system at time $t = 0$ is $\mathbf{s}$, *i.e.*, $\phi(0, \mathbf{s}) = \mathbf{s}$, then the state of this system at time t is $\phi(t, \mathbf{s})$.

Consider now the parameters in equations 1.9 and 1.8 that determine the differential equations. They consist of quantities some of which are deterministic like $\Lambda(\mathbf{q})$ but not known precisely a-priori, and some are non-deterministic like the values of the state $\mathbf{q}$, $\dot{\mathbf{q}}$, and the measured reaction force $J^T(\mathbf{q})\lambda$ used in the feedback control law.

1.3 Review of Previous Work

A substantive description of this formalism is given by Latombe [Lat 91]. At the expense of repetition, a brief overview of robotics related results are presented here. A review of work related to control uncertainty is presented separately in Chapter 3.

Lozano-Pérez, Mason, Taylor [LMT 84] first presented a formal framework of synthesizing Fine Motion Plans in the context of compliant motion strategies for assembly of parts. This paper considers uncertainties broadly divided into *Control Uncertainty*, *Sensing Uncertainty*, and *Model/Shape Uncertainty*. It prescribed bounds on each of the uncertainties and proposed notions of strongly guaranteed and weakly guaranteed plans in a recursive structure of motion commands and associated termination and selection predicates. This formalism with provable properties to deal with uncertainty was in sharp contrast with the prevalent *Skeleton Refinement* method of Taylor [Tay 76] and Lozano-Pérez [Loz 76] and concurrently appearing paradigm of *Inductive Learning* by Dufay and Latombe [DufLat 84].

Mason [Mas 84] considered *Bounded-Complete Strongly Guaranteed Plans* with continuous sensing history, though his completeness result should be seen in light of our observation that the interpretation of sensing measurements is also affected by the termination condition. Erdmann [Erd 84] separated *goal attainment recognition* and

goal reachability. A subset of the goal called *Kernel* [Erd 84, Lat 88] is constructed where it is possible to recognize the achievement of goal. Then, a *backprojection* of the kernel is constructed which are those points for which all trajectories for the given control model reach the kernel. An algorithm in *mini-world* called *Backprojection from Kernel* was given by Latombe [Lat 88]. Donald [Don 87] considered weakly guaranteed plans with additional structure for Error Detection and Recovery. Lastly, a forthcoming paper by Brost and Christiansen [BrCh] considers *Probabilistic Plans* with a probabilistic likelihood of success.

In fine motion planning, among descriptions of the moving object and the obstacles polyhedral, polygonal, and objects bounded by patches of algebraic surfaces are common. A typical configuration space is the Euclidean Plane for one rigid body translating in a plane - this is the space considered in all planners except those of Buckley [Buck 86], Canny [Canny 89], and Brost [Brost 92]. Other typical configuration spaces are $SE(3)$ - the group of rigid body transformations in physical three space, and T^n - the n torus describing configurations of an n-revolute jointed manipulator.

Among control schemes that model the motion, there are first-order approximations and exact second-order differential equations. A constant first-order model known as *velocity control* in the free space and *damper control* in both the free and contact space was used in the original Lozano-Pérez, Mason, Taylor paper and has been used subsequently in the planners of Donald [Don 87], Briggs [Brig 89], Latombe [Lat 88], Latombe, Lazanas, and Shekhar [LLS 91], Brost [Brost 92], and Lazanas and Latombe [LazLat 92]. A linear first-order model which is an approximation of the commonly used *spring dynamics* was used by Buckley [Buck 86]. There exists no fine motion planner that uses a second-order model, except perhaps an a-priori model of the envelope of trajectories used by Canny [Canny 89]. Force control literature [Whit 86], however, contains several such potential models. Among them is the model, used by Khatib [Khat 87], of a *unit mass* moving in the operational space with position derivative and proportional-derivative force feedback.

Along the dimension of sensing, examples are position sensing, force sensing, and time sensing. These are direct sensing. Indirect sensing include camera based vision

sensing, sparse beam type sensing, and models thereof, particularly the *perceptual equivalence class* of Donald and Jennings [DonJen 91]. Some examples of a termination predicate are the *Sticking Termination* used by Donald [Don 87], the position, force and time based predicates used by Canny [Canny 89], and the position and force sensing based termination predicates used by Latombe [Lat 88], and Latombe, Lazanas, and Shekhar [LLS 91].

Natarajan [Nat 88] first showed that Strongly Guaranteed Fine Motion Planning is P-Space hard. Several other general results on the complexity are by Canny and Reif, and Canny. Checking if a point lies in the time dependent forward projection is NP-hard [CanRei 87]. A three-dimensional compliant motion planning problem is non-deterministic exponential time hard [CanRei 87]. An algorithm to find an n-step plan is given by Canny [Canny 89] that is double exponential in the size of the environment, the number of steps, n, in the plan, and the dimension of the space. A less general algorithm given by Donald [Don 87] is single exponential in the number of steps. Several variants of One-Step Strongly Guaranteed Fine Motion Planner in planar configuration space with a damper model of control uncertainty have low complexity. They are the $n\log(n)$ algorithm of Friedman, Hershberger, and Snoeyink [FHS 89], the $n^2\log(n)$ of Briggs [Brig 89], and the $n\log(n)$ of Latombe, Lazanas, and Shekhar [LLS 91]. A recently proposed multi-step strongly guaranteed fine motion planner by Lazanas and Latombe [LazLat 92] in planar configuration space with velocity control was given with a polynomial complexity under the assumption that sensing and control are perfect inside landmark regions, while sensing is inexistent and control is imperfect outside such regions.

1.4 Overview of the Rest of the Thesis

Chapter 2 expands further on the results in Section 1.1.2 on One-Step Fine Motion Planning with uncertainty in control and sensing. We also construct some simple examples of One-Step Preimages.

Chapter 3 considers the problem of characterizing control uncertainty. We give a brief overview of previous results from Differential Inclusion, Hamiltonian systems,

and Hamilton-Jacobi Theorem. The remaining parts of this chapter give motivation, constructions and the proof of all the new results on characterizing the control uncertainty. We end this chapter by constructing two simple examples. The first example is in a simple two-dimensional domain and illustrates Theorem 1.2. The second example is in a three-dimensional domain. It considers some degenerate cases of Theorem 1.2. However, a natural generalization of this example gives a well defined boundary of the forward projection of the stable and unstable manifolds in all dimensions.

Chapter 2

Fine Motion Planning Problem

In this chapter[1], we present a proposal of the fine motion planning problem. The inputs to a Fine Motion Planner consist of Geometric Models, Models of Uncertainties, and an Initial, and a Goal Subset. The model of uncertainties comprises control uncertainty, sensing uncertainty, and uncertainty in the geometric model. We do not consider here the uncertainty in the geometric model. We give a preliminary construction of the configuration space and the configuration space obstacles for a rigid body motion in the presence of other stationary rigid body obstacles. Equations of rigid body motion transform smoothly with any choice of coordinate system on the physical space. We give such transformations for second-order equations in Section 2.1.2. We pose the problem of control uncertainty on smooth sets in Section 2.2 cast as a problem in differential inclusion and define the set of solutions that describe control uncertainty. Section 2.3 presents a model of sensing uncertainty as a set valued map - particularly position, velocity, and force sensing. We, then, define a *One-Step Preimage* - a building block of any complete planner in this framework. We end this chapter by constructing some simple preimages that serve as examples, including one where embedding the knowledge of the termination condition in the construction of preimage augments its size.

[1] A Part of this chapter has benefited from suggestions by Randy Wilson.

2.1 Configuration Space

The problem of rigid objects moving amongst other objects can be reduced to the problem of a point moving in the configuration space. In this framework, corresponding to a motion in free space, there is a motion of the point in the free space of configuration space, and corresponding to the motion of the moving object when it touches one or several of the objects in the workspace, there is a motion of the point on the boundaries of the *Configuration Space Obstacles.* Let $\mathcal{S}$ be either the two-dimensional or three-dimensional physical space. Consider $\mathcal{S}$ as the *standard Euclidean space* $\mathbf{E}^n$ where $n(= 2\, or\, 3)$ is the dimension of the physical space $\mathcal{S}$. Consider rigid bodies $\mathcal{A}_i$, $i = 1, \ldots, r$, as *compact, closed* and *regular* [HopWil 86] subsets of the space $\mathcal{S}$ at the *initial configuration.* The $\{\mathcal{A}_i,\, i = 1, \ldots, r\}$'s are moving objects. Displacements of each rigid body $\mathcal{A}_i$ extends uniquely to isometries of $\mathcal{S}$. The set of *isometries* form a *Special Orthogonal Lie group* denoted by $SE(n)$. The elements $\mathbf{q} \in SE(n)$ are maps

$$\mathbf{q}: \mathcal{S} \rightarrow \mathcal{S} \tag{2.1}$$

that assign to each point $\mathbf{p} \in \mathcal{S}$ a unique point $\mathbf{p}' = \mathbf{q}(\mathbf{p}) \in \mathcal{S}$. If there is one free-flying rigid object in a planar workspace capable of translating and rotating, then an element of $SE(2)$ specifies a rigid body displacement. Note that $SE(2)$ is the same as the interior of a solid torus $\mathbf{R}^2 \times S^1$. Now consider several rigid bodies. If there are r independent moving bodies, r copies of $SE(n)$ specify possible displacements of the rigid bodies. Effectively, an element in $\prod_r SE(n)$ uniquely determines the position and orientation of all moving bodies. Any kinematic relationship among the moving bodies specifies a subset of the product space $\prod_r SE(n)$. For example, if there is a manipulator with six revolute joints in three space, elements on the surface of a six torus $\mathbf{T}^6 \subset \prod_6 SE(3)$ determine the position and orientation of all the links. If there is a mobile robot translating and rotating on the floor sharing the workspace with a manipulator with six revolute joints, then elements on the surface of a six torus and interior of a solid torus ($\equiv \mathbf{T}^7 \times \mathbf{R}^2$) specify all possible configurations.

Define the *Configuration Space*, $\mathcal{Q}$, of several rigid objects or several interconnected groups of rigid objects to be the set of all isometries that uniquely determine

the position and orientation of the objects in the physical space $\mathcal{S}$. For any $\mathbf{q} \in \mathcal{Q}$, denote by $\mathcal{A}_i(\mathbf{q})$ the displaced rigid body $\mathcal{A}_i$ at the configuration $\mathbf{q} \in \mathcal{Q}$. We also use $\mathcal{A}(\mathbf{q})$ to denote all moving bodies $i = 1, \ldots, r$, at the configuration $\mathbf{q}$. Let $e \in \mathcal{Q}$ denote the identity map. Then, $\mathcal{A}_i(e)$ represents the moving object at the *initial configuration* - the subset $\mathcal{A}_i$ of $\mathcal{S}$.

In addition to the moving bodies in the physical space $\mathcal{S}$, consider other stationary rigid bodies called *obstacles* $\mathcal{A}_i$, $i = r+1, \ldots, r+s$, that are *disjoint*, *closed*, and *regular* but not necessarily *compact* subsets of $\mathcal{S}$. Since these objects are stationary, for notational simplicity, we define $\mathcal{A}_i(\mathbf{q}) \equiv \mathcal{A}_i, i = r+1, r+s$. Define the *Configuration Space Obstacles* as the set of points, $\mathbf{q}$, in the configuration space $\mathcal{Q}$ where the bodies $\mathcal{A}(\mathbf{q})$ overlap each other, *i.e.*, $\mathcal{Q}_{obst} = \cup_{i=1,r}\{q \in \mathcal{Q} | \mathcal{A}_i(\mathbf{q}) \cap \mathcal{A}_j(\mathbf{q}) \neq \emptyset,\, j = 1, \ldots, r+s,\, j \neq i\}$. The complement of the Configuration space obstacles in the configuration space is the *free space*, *i.e.*, $\mathcal{Q}_{free} = \mathcal{Q} \setminus \mathcal{Q}_{obst}$. Define *Contact C-space* as $\mathcal{Q}_{contact} = \{\mathbf{q} \in \mathcal{Q} | \mathcal{A}_i(\mathbf{q}) \cap \mathcal{A}_j(\mathbf{q}) \neq \emptyset, int(\mathcal{A}_i(\mathbf{q})) \cap int(\mathcal{A}_j) = \emptyset\,, i, j = 1, \ldots, r+s,\, i \neq j\}$.

Definition 2.1 *Define Valid C-space as*

$$\mathcal{Q}_{valid} = \mathcal{Q}_{free} \cup \mathcal{Q}_{contact}.$$

Note that since the moving objects $\mathcal{A}_i, i = 1, \ldots, r$ are compact, closed, and regular, and the stationary objects $\mathcal{A}_i, i = r+1, \ldots, r+s$ are closed, and regular, the space $\mathcal{Q}_{valid}$ is closed (see Latombe [Lat 91]) , but it may not be regular (see example in Hopcroft and Wilfong [HopWil 86]). In addition to the assumptions that make the valid configuration space a closed subset of configuration space, we assume that it is also bounded. Effectively, we only consider compact valid configuration spaces. For example, the configuration space of a planar rigid body translating in the plane without any obstacles is not compact. By assumption, we limit our interest to a compact subset of the configuration space. Several configuration spaces like that of the n-revolute jointed manipulator satisfy this condition naturally.

Consider the case when the configuration space $\mathcal{Q}$ is a subset of a smooth manifold and the valid C-space, $\mathcal{Q}_{valid}$, is a piecewise smooth subset of the configuration space. In most applications of robotics, the configuration space is an algebraic manifold (defined by finitely many polynomial equations) and the valid C-space is

a semi-algebraic subset of the algebraic manifold - see for example Schwartz and Sharir [SS 83b], Canny [Canny 88] and Ge and McCarthy [GeMc 90]. A *stratification* $\underline{\underline{\mathcal{Q}}}_{valid}$ of the valid C-space is a partition of $\mathcal{Q}_{valid}$ into a disjoint union of a locally finite number of subsets $\mathcal{S}_i$, each of constant dimension called a *stratum* such that the frontier of each stratum is the union of lower dimensional strata. Consider each stratum $\mathcal{S}_i$ as a subset of a smooth manifold M_i. For this purpose, it is sufficient to consider the existence proof by Hironaka [Shaf 74] of a projective non-singular model of every irreducible, projective and smooth variety of an arbitrary dimension over characteristic zero fields.

2.1.1 Tangent Cone and Bundle of Valid C-Space

Consider the triple $\pi = (T\mathcal{Q}, \mathcal{Q}, \pi)$ called the *tangent bundle* of the configuration space $\mathcal{Q}$ where the map $\pi: T\mathcal{Q} \to \mathcal{Q}$ is the natural projection defined as $\pi(\mathbf{q}, \mathbf{v_q}) = \mathbf{q}$. There exists an atlas $\{(U_i, \phi_i)\}$ of $\mathcal{Q}$ and a collection of mappings $\{\psi_i\}$ satisfying:

- $\psi_i: \pi^{-1}U_i \to \phi_i U_i \times \mathbf{R}^n$ is a local trivialization of the tangent bundle,
- $\{(\pi^{-1}U_i, \psi_i)\}$ is an atlas of $T\mathcal{Q}$, and
- for all i, j, $(\psi_j \circ \psi_i^{-1}, \phi_j \circ \phi_i^{-1})$ is a C^r local tangent bundle map.

The charts $(\pi^{-1}U_i, \psi_i)$ are local tangent bundle charts. Denote the local tangent bundle charts by the triple (U_i, ϕ_i, ψ_i). The space $T\mathcal{Q}$ is called the *total space*, $\mathcal{Q}$ the base space, and $\pi^{-1}(\mathbf{q})$ for $\mathbf{q} \in \mathcal{Q}$, the fibre over $\mathbf{q}$. Often we will denote elements $(\mathbf{q}, \mathbf{v_q}) \in T\mathcal{Q}$ as $\mathbf{s}$ - the state, so that $\pi(\mathbf{s}) = \mathbf{q}$ as usual. Denote $\pi_2(\mathbf{s}) = \mathbf{v_q}$, the projection onto the second factor, though this projection will be used less often.

Define the *restriction tangent bundle* $\pi = (T\mathcal{Q}|_{\mathcal{Q}_{valid}}, \mathcal{Q}_{valid}, \pi)$ of the valid configuration space as the bundle $\bigcup_{\mathbf{q}\in\mathcal{Q}_{valid}} T_{\mathbf{q}}\mathcal{Q}$. This is just the restriction of the tangent bundle of the configuration space to the valid configuration space. To construct *the tangent bundle of the valid configuration space*, first construct the tangent bundle to a stratum $\mathcal{S}_i$ denoted $\pi_i = (T\mathcal{S}_i, \mathcal{S}_i, \pi_i)$. Construct the tangent bundle of the valid configuration space, $\pi = (T\mathcal{Q}_{valid}, \mathcal{Q}_{valid}, \pi)$, to be the bundle $\bigcup_i T\mathcal{S}_i$. Note the difference between the restriction tangent space $T\mathcal{Q}|_{\mathcal{S}_i}$ and the tangent space of the strata

considered as a manifold $T\mathcal{S}_i$. For a point in the strata, say $\mathcal{S}_j$, corresponding to the free space, the dimension of $T\mathcal{S}_j$ is same as the $T\mathcal{Q}|_{\mathcal{S}_j}$ and for a point in a strata, say $\mathcal{S}_k$, of dimension $n-p$, where n is the dimension of $\mathcal{Q}$ and p is the codimension of the strata $\mathcal{S}_k$, the dimension of $T\mathcal{S}_k$ is $2(n-p)$ whereas the dimension of $T\mathcal{Q}|_{\mathcal{S}_k}$ is $2n-p$.

Parallel to the construction of tangent cones to analytic varieties [Wh 65b], we give a definition of the tangent cones to real semi-algebraic set. In particular, we want to capture the notion of the set of all tangents to a point in the valid configuration space. Let the tangent cone to a point $\mathbf{q} \in \mathcal{Q}_{valid}$ be denoted $\mathbf{C_q}\mathcal{Q}_{valid}$. The tangent cone consists of those $\mathbf{v_q} \in T_\mathbf{q}\mathcal{Q}|_{\mathcal{Q}_{valid}}$ such that there exists a sequence of points $\{\mathbf{q}_i\} \rightarrow \mathbf{q} \in \mathcal{Q}_{valid}$, a sequence of positive real numbers $\{a_i\}, a_i > 0$, and $a_i(\mathbf{q}_i - \mathbf{q}) \rightarrow \mathbf{v_q}$. The tangent cone consists of limits of secants from $\mathbf{q}$ to points in $\mathcal{Q}_{valid}$. For analytic varieties, Whitney [Wh 65b] shows that the vanishing of the first homogeneous part of the defining polynomial of the variety defines the tangent cone. We are not aware of a similar constructive definition of the tangent cones to real semi-algebraic sets. Tangent cones in general are not vector spaces even for algebraic sets. So, when we piece the tangent cone at every point in the semi-algebraic valid configuration space, instead of a vector bundle, we get a fiber bundle. Denote the *Tangent Cone* fiber bundle as $\mathbf{C}\mathcal{Q}_{valid}$.

The tangent space of a point in the valid configuration space is a subset of the tangent cone at the point of the valid configuration space, which itself is a subset of the restriction tangent space of the valid configuration space:

$$T_\mathbf{q}\mathcal{Q}_{valid} \subset \mathbf{C_q}\mathcal{Q}_{valid} \subset T_\mathbf{q}\mathcal{Q}. \tag{2.2}$$

The containment of the tangent cone in the restriction tangent space is by construction. To see that the tangent cone may be a proper subset, consider the semi-algebraic set in the plane defined by $x \geq 0$ and consider any point $(0, y)$ in this set. The tangent cone at this point is the set $\mathbf{v}_{(0,y)} = \{v_x \geq 0\}$, whereas the restriction tangent space at this point is the plane $\mathbf{R}^2$ itself. Now consider the containment of the tangent bundle of the valid configuration space in the tangent cone fiber bundle. Consider the tangent space $T_\mathbf{q}\mathcal{S}_j$ of a point $\mathbf{q} \in \mathcal{S}_j$. If in the definition of tangent cone, the sequence

$\{\mathbf{q}_i\}$ is restricted to the strata $\mathcal{S}_j$, the tangent cone $\mathbf{C_q}\mathcal{Q}_{valid}$ becomes identical to the tangent space $T_{\mathbf{q}}\mathcal{S}_j$.

2.1.2 Representation of Configuration Space

Assuming that the *initial configuration* of the moving object in the physical space denoted $e \in SE(n)$ is given, the configuration space obstacles are described independent of any representation of $SE(n)$. However, various choices of coordinate system on the physical space yield naturally a representation of the configuration space and the configuration space obstacles. These transformations for the representation of $SE(n)$ and its first tangent bundle are given by Lončarić [Lonc 87]. In addition to them, we establish transformations for the elements in the second tangent bundle - necessary for the equations of motion that are second-order. Classically, such transformations arise from the action of a *Lie group* on a space - in our case the action of *Special Orthogonal Lie Group* on the physical space which is a two or three-dimensional Euclidean space. As a consequence of the transformations, we also prove that the configuration space (and their subsets) transform diffeomorphically (smoothly) due to a change in the choice of coordinate. Since smooth transformations preserve the differential structure such as the phase-space flow, and the type of singularities of vector fields, it is sufficient to consider vector fields on the configuration space and the obstacles with an arbitrary choice of coordinate system on the physical space.

In addition to this structure, any other initial arrangement of the moving objects in the physical space yields an intrinsically different representation of configuration space obstacles. The representation of configuration space obstacles for any new initial arrangement is related to a previous one by *right translation* map $b \overset{R_a}{\mapsto} ba$ for $a, b \in SE(n)$, where a is the map that sends one initial configuration of the moving objects to the other. These transformations are also smooth maps and are treated extensively in books such as Abraham and Marsden [AbMa 78].

Special Orthogonal Lie Group

By a *choice of the coordinate system* on the physical space $\mathcal{S}$ we mean a norm-preserving orthogonal map

$$\mathbf{x}: \mathcal{S} \rightarrow \mathbf{R}^n : \mathbf{p} \overset{\mathbf{x}}{\mapsto} (x^1(\mathbf{p}), x^2(\mathbf{p}), \ldots, x^n(\mathbf{p})) \tag{2.3}$$

where $\mathbf{R}^n$ is n-tuple of the set of reals, each x^i is a scalar map from the physical space $\mathcal{S}$ to the set of reals $\mathbf{R}$.

The configuration space of one free-flying rigid body in space $\mathcal{S}$ is $SE(n)$. For a choice of coordinate system $\mathbf{x}$ on $\mathcal{S}$, an element $\mathbf{q}$ of $SE(n)$ has several prevalent representations including *homogeneous transformation matrix*, *Plüker screw coordinates*, and *dual quaternions* [BotRot 79]. Denote the representation of an element $\mathbf{q} \in SE(n)$ by $\mathbf{q}^{\mathbf{x}}$ with the choice of coordinate system $\mathbf{x}$ on $\mathcal{S}$. As an example, the representation of an element $\mathbf{q} \in SE(n)$ in the homogeneous transformation matrix is

$$\mathbf{q}^{\mathbf{x}} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \tag{2.4}$$

where $\mathbf{R}$ is an $n \times n$ rotation matrix and $\mathbf{t}$ is a translation n vector, so that $\mathbf{q}(\mathbf{p}) = \mathbf{x}^{-1}(\mathbf{R}\mathbf{x}(\mathbf{p}) + \mathbf{t})$. For an arbitrary representation, the identity $\mathbf{q}(\mathbf{p}) = \mathbf{x}^{-1} \circ \mathbf{q}^{\mathbf{x}} \circ \mathbf{x}(\mathbf{p})$ represents map $\mathbf{q}$ in terms of its representation $\mathbf{q}^{\mathbf{x}}$ and the coordinate system map $\mathbf{x}$. Note the converse identity $\mathbf{q}^{\mathbf{x}}(\mathbf{p}) = \mathbf{x} \circ \mathbf{q} \circ \mathbf{x}^{-1}(\mathbf{p})$.

Another choice of coordinate system, say $\mathbf{y}$, changes the representation of elements $\mathbf{q}$ of $SE(n)$. If the two choices of coordinate system $\mathbf{x}$ and $\mathbf{y}$ are related by a map $\mathbf{h} \in SE(n)$, such that $\mathbf{y} = \mathbf{x} \circ \mathbf{h}^{-1}$, then two representations of any element $\mathbf{q} \in SE(n)$ are related by:

$$\begin{aligned} \mathbf{q}^{\mathbf{y}} &= \mathbf{y} \circ \mathbf{q} \circ \mathbf{y}^{-1} \\ &= \mathbf{x} \circ \mathbf{h}^{-1} \circ \mathbf{x}^{-1} \circ \mathbf{q}^{\mathbf{x}} \circ \mathbf{x} \circ \mathbf{h} \circ \mathbf{x}^{-1} \\ &= (\mathbf{h}^{-1})^{\mathbf{x}} \circ \mathbf{q}^{\mathbf{x}} \circ \mathbf{h}^{x}. \end{aligned} \tag{2.5}$$

Tangent Space of Special Orthogonal Lie Group

The tangent space $T_e SE(n)$ to the special orthogonal Lie group at the identity $e \in SE(n)$ (the initial configuration), denoted $se(n)$, is the space of infinitesimal rigid

body motions. A choice of coordinate system $\mathbf{x}$ on $\mathcal{S}$ also determines a basis of the vector space $se(n)$. For example, denote the representation of an element $\mathbf{v}_e \in se(3)$ by

$$\mathbf{v}_e^{\mathbf{x}} = [\,\omega_{e1}^{\mathbf{x}} \quad \omega_{e2}^{\mathbf{x}} \quad \omega_{e3}^{\mathbf{x}} \quad v_{e1}^{\mathbf{x}} \quad v_{e3}^{\mathbf{x}} \quad v_{e3}^{\mathbf{x}}\,]^T \tag{2.6}$$

where $\omega_e^{\mathbf{x}} = [\,\omega_{e1}^{\mathbf{x}} \quad \omega_{e2}^{\mathbf{x}} \quad \omega_{e3}^{\mathbf{x}}\,]^T$ is the angular velocity and $[\,v_{e1}^{\mathbf{x}} \quad v_{e2}^{\mathbf{x}} \quad v_{e3}^{\mathbf{x}}\,]^T$ is the translational velocity of the origin of coordinate system $\mathbf{x}$, both expressed in $\mathbf{x}$. Another choice of coordinate system, say $\mathbf{y}$, on the space $\mathcal{S}$ induces a map of an element $\mathbf{v}_e \in se(n)$ from one represenation to the other. First, consider for a given $\mathbf{h} \in SE(n)$, the conjugation map

$$\mathbf{q} \mapsto \mathbf{h}\mathbf{q}\mathbf{h}^{-1}. \tag{2.7}$$

Define the adjoint $Ad_{\mathbf{h}}$ to be the derivative of the conjugation map at $\mathbf{q} = e$. The adjoint is a map $Ad_{\mathbf{h}}\colon se(n) \longrightarrow se(n)$. For example, if $\mathbf{h}$ is represented as a homogeneous transformation matrix in the coordinate system $\mathbf{x}$ with $\mathbf{R}^{\mathbf{h}^{\mathbf{x}}}$ as the rotation matrix and $\mathbf{t}^{\mathbf{h}^{\mathbf{x}}}$ as the translation vector, then the adjoint $Ad_{\mathbf{h}}$ is represented as

$$Ad_{\mathbf{h}}^{\mathbf{x}} = \begin{bmatrix} \mathbf{R}^{\mathbf{h}^{\mathbf{x}}} & 0 \\ [\mathbf{t}^{\mathbf{h}^{\mathbf{x}}}]\mathbf{R}^{\mathbf{h}^{\mathbf{x}}} & \mathbf{R}^{\mathbf{h}^{\mathbf{x}}} \end{bmatrix} \tag{2.8}$$

where $[\mathbf{t}^{\mathbf{h}^{\mathbf{x}}}]$ represents the cross-product operator of the vector $\mathbf{t}^{\mathbf{h}^{\mathbf{x}}}$ defined as

$$[\mathbf{t}] = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}. \tag{2.9}$$

Now, consider the representations of $\mathbf{v}_e$ in coordinate system $\mathbf{x}$ and $\mathbf{y}$. The derivative of equation 2.5 at the identity is the required map between the two represenations. Thus,

$$\mathbf{v}_e^{\mathbf{y}} = Ad_{\mathbf{h}^{-1}}^{\mathbf{x}}(\mathbf{v}_e^{\mathbf{x}}). \tag{2.10}$$

The equation 2.10 gives the mapping of elements of the tangent space of the configuration space $se(n)$ at the identity e - a relation that transforms velocity given in the $\mathbf{x}$ frame to the velocity in the $\mathbf{y}$ frame when the object is at the initial configuration. The mapping of velocity in frame $\mathbf{x}$ to that in $\mathbf{y}$ when the object is at an arbitrary configuration $\mathbf{q}$ is related by another mapping. In effect, we would like to construct

a representation of $TSE(n)$. Since $SE(n)$ is a Lie group, its tangent bundle is parallelizable (trivial), the tangent bundle $TSE(n)$ can be identified with $SE(n) \times se(n)$. We derive the transformation from one coordinate system $\mathbf{x}$ to another $\mathbf{y}$ in terms of the adjoint in equation 2.8 and the map corresponding to the configuration $\mathbf{q}$.

Consider coordinate systems $\mathbf{z}$ and $\mathbf{z}'$ related to frames $\mathbf{x}$ and $\mathbf{y}$ by

$$\begin{aligned} \mathbf{z} &= \mathbf{x} \circ \mathbf{q}^{-1}, \\ \text{and} \qquad \mathbf{z}' &= \mathbf{y} \circ \mathbf{q}^{-1}. \end{aligned}$$

They represent the rigid body displacements of the frame $\mathbf{x}$ and frame $\mathbf{y}$ by the map $\mathbf{q} \in SE(n)$. In other words, frames $\mathbf{z}$ and $\mathbf{z}'$ are the displaced configurations. The relations $\mathbf{v}_e^{\mathbf{z}'} = \mathrm{Diag}(\mathbf{R}^{\mathbf{q}^{-1}\mathbf{x}}, 2 \times n)\mathbf{v}_{\mathbf{q}}^{\mathbf{y}}$, and $\mathbf{v}_e^{\mathbf{z}} = \mathrm{Diag}(\mathbf{R}^{\mathbf{q}^{-1}\mathbf{x}}, 2 \times n)\mathbf{v}_{\mathbf{q}}^{\mathbf{x}}$ follows where $\mathrm{Diag}(\mathbf{R}, i \times n)$ is i blocks of $n \times n$ $\mathbf{R}$ matrix on the diagonal. Also,

$$\begin{aligned} \mathbf{z}' &= \mathbf{z} \circ \mathbf{q} \circ \mathbf{h}^{-1} \circ \mathbf{q}^{-1} \\ &= \mathbf{q}^{\mathbf{z}} \circ (\mathbf{h}^{-1})^{\mathbf{z}} \circ (\mathbf{q}^{-1})^{\mathbf{z}} \circ \mathbf{z}. \end{aligned}$$

It is easy to verify that $\mathbf{q}^{\mathbf{z}} = \mathbf{q}^{\mathbf{x}}$, $(\mathbf{h}^{-1})^{\mathbf{z}} = (\mathbf{q}^{-1})^{\mathbf{x}} \circ (\mathbf{h}^{-1})^{\mathbf{x}} \circ \mathbf{q}^{\mathbf{x}}$ and $(\mathbf{q}^{-1})^{\mathbf{z}} = (\mathbf{q}^{-1})^{\mathbf{x}}$, so that $\mathbf{z}' = (\mathbf{h}^{-1})^{\mathbf{x}} \circ \mathbf{z}$ implying $\mathbf{v}_e^{\mathbf{z}'} = Ad^{\mathbf{x}}_{\mathbf{h}^{-1}}(\mathbf{v}_e^{\mathbf{z}})$. These relate the velocities of frames $\mathbf{z}$ and $\mathbf{z}'$. Therefore, the velocities $\mathbf{v}_{\mathbf{q}}^{\mathbf{y}}$ and $\mathbf{v}_{\mathbf{q}}^{\mathbf{x}}$ are related by

$$\mathbf{v}_{\mathbf{q}}^{\mathbf{y}} = \mathrm{Diag}(\mathbf{R}^{\mathbf{q}^{\mathbf{x}}}, 2 \times n)\; Ad^{\mathbf{x}}_{\mathbf{h}^{-1}}\; \mathrm{Diag}(\mathbf{R}^{\mathbf{q}^{-1}\mathbf{x}}, 2 \times n)\; \mathbf{v}_{\mathbf{q}}^{\mathbf{x}}. \tag{2.11}$$

Second Tangent Bundle of SE(n)

The space $Tse(n)$, the double tangent space at the identity of $SE(n)$, also acquires a basis with the choice of the coordinate system $\mathbf{x}$. Denote a representation of an element $\beta_{\mathbf{v}_e} \in Tse(n)$ in this choice as

$$\beta^{\mathbf{x}}_{\mathbf{v}_e} = \begin{bmatrix} \mathbf{a}^{\mathbf{x}}_{\mathbf{v}_e} \\ \mathbf{w}^{\mathbf{x}}_{\mathbf{v}_e} \end{bmatrix}, \tag{2.12}$$

where $\mathbf{a}^{\mathbf{x}}_{\mathbf{v}_e}$ is the acceleration vector, and $\mathbf{w}^{\mathbf{x}}_{\mathbf{v}_e}$ is the velocity vector corresponding to the derivative of the base space vector $e \in SE(n)$. For $n = 3$, $\beta^{\mathbf{x}}_{\mathbf{v}_e}$ has twelve components. Another choice of coordinate system $\mathbf{y}$ on the space $\mathcal{S}$ induces a map

of an element $\beta_{\mathbf{v}_e} \in Tse(n)$ from one representation to another. We establish the following transformation

$$\beta^{\mathbf{y}}_{\mathbf{v}_e} = \begin{bmatrix} Ad^{\mathbf{x}}_{\mathbf{h}^{-1}} & \Lambda^{\mathbf{x}}_{\mathbf{h}^{-1}} \\ 0 & Ad^{\mathbf{x}}_{\mathbf{h}^{-1}} \end{bmatrix} \beta^{\mathbf{x}}_{\mathbf{v}_e}, \tag{2.13}$$

where

$$\Lambda^{\mathbf{x}}_{\mathbf{h}^{-1}} = \begin{bmatrix} 0 & 0 \\ -[[t]\mathbf{R}^{\mathbf{h}^{-1}\mathbf{x}}\omega^{\mathbf{x}}]\mathbf{R}^{\mathbf{h}^{-1}\mathbf{x}} & 0 \end{bmatrix}. \tag{2.14}$$

When the rotation part of the map $\mathbf{h}$ is an identity, this transformation is equivalent to the classical kinematics expression $\mathbf{a}_p = \mathbf{a}_o + \alpha \times \mathbf{r}_{op} + \omega \times (\omega \times \mathbf{r}_{op})$ for the acceleration of another point p of a rigid body when the accleration of point o is given. This relation between accelerations in two choices of coordinate systems holds for any velocity $\mathbf{v}_e$ at the identity configuration. At an arbitrary state $(\mathbf{q}, \mathbf{v_q})$, the relation

$$\beta^{\mathbf{y}}_{\mathbf{v_q}} = \mathrm{Diag}(\mathbf{R}^{\mathbf{q}^{\mathbf{x}}}, 4 \times n) \begin{bmatrix} Ad^{\mathbf{x}}_{\mathbf{h}^{-1}} & \Lambda^{\mathbf{x}}_{\mathbf{h}^{-1}} \\ 0 & Ad^{\mathbf{x}}_{\mathbf{h}^{-1}} \end{bmatrix} \mathrm{Diag}(\mathbf{R}^{\mathbf{q}^{-1}\mathbf{x}}, 4 \times n)\, \beta^{\mathbf{x}}_{\mathbf{v_q}} \tag{2.15}$$

parallel to the one given in equation 2.11 that relates velocities, gives the transformation of the elements $\beta_{\mathbf{v_q}} \in T_{\mathbf{v_q}}(TSE(n))$.

Diffeomorphism of Configuration Space

Proposition 2.1 *Any choice of coordinate system induces diffeomorphic representation of the Special Orthogonal Lie group $SE(n)$.*

Proof: The equations 2.5 and 2.11 give explicit maps relating any two representations of $SE(n)$ and their derivatives. They are non-singular. Any two choices are thus, homeomorphisms of $SE(n)$. Infact these are analytic maps. So, different choices of coordinate systems induce smooth diffeomorphic representations of the Lie Group SE(n).$\diamond$

Any vector field on $SE(n)$ and tangent bundle of $SE(n)$ have a diffeomorphic representation for any choice of the coordinate system and any two such representations are related according to equations 2.11 and 2.15.

Now, consider smooth manifolds M that are subsets of $SE(n)$. Any vector field on TM transforms diffeomorphically from one coordinate system to another. Consider the case of several rigid bodies. The product space $\prod_r SE(n)$ has a smooth diffeomorphic representation for each choice of coordinate systems. Any subset of the product configuration space also transforms nicely; for instance consider various representations of the six-torus as a subset of $\prod_6 SE(3)$ for various choices of a coordinate system.

2.1.3 Forces in Configuration Space

Forces are elements of the cotangent bundle T^*Q of the configuration space. A force element $\mathbf{f_q}$ at $\mathbf{q}$ is a linear function on the tangent space, *i.e.,* $\mathbf{f_q}: T_\mathbf{q}Q \rightarrow \mathbf{R}$. Intuitively, a force $\mathbf{f_q}$ at $\mathbf{q}$ maps an infinitesimal displacement $\mathbf{v_q} \in T_\mathbf{q}Q$ to the reals, the work done, also called *Virtual Work*. Now, consider an example of a force field on the configuration space specified by a potential function $V: Q \rightarrow \mathbf{R}$. Define the covariant section $dV: Q \rightarrow T^*Q$ of the cotangent bundle such that $dV(\mathbf{q})(X(\mathbf{q})) = X(\mathbf{q})(V(\mathbf{q}))$ for $X: Q \rightarrow TQ$, a vector field on the configuration space. The covariant section dV defines a force field on the configuration space. When configuration space is the space of rigid body motions in three-dimensional space, the forces consist of the usual three moments and three forces. Another example is the set of six joint torques of a manipulator with six revolute joints. In classical mechanics, such forces are called *generalized forces.* We use the term *generalized forces* to denote the forces in the configuration space.

Contrary to a natural basis that exists for elements in the tangent bundle, there is no natural choice of basis for the cotangent bundle. Given a Riemannian metric on the manifold, however, there exists a unique representation of the elements in the cotangent bundle. Consider a tensor $g \in T_2^0(Q)$, representing a metric on the configuration space. Then, there exists an isomorphism $g^\sharp \in L(T^*M, TM)$ of the cotangent and tangent bundle defined as $\mathbf{f}(\mathbf{v}) = g(g^\sharp(\mathbf{f}), \mathbf{v})$ for $\mathbf{f} \in T^*M$ and $\mathbf{v} \in TM$.

Now, consider forces on subsets of the configuration space. In particular, consider manifolds $M \subset Q$, either a closed subset of codimension p or an open subset. Forces on M are elements of the cotangent bundle of the configuration space restricted to

the manifold M. So, a force $\mathbf{f}_\mathbf{q}$ at a point $\mathbf{q}$ in M in $T^*_\mathbf{q}\mathcal{Q}$ is a linear function on the tangent space of configuration space restricted to the manifold M *i.e.* $\mathbf{f}_\mathbf{q}: T_\mathbf{q}\mathcal{Q} \rightarrow \mathbf{R}$. In this way, specification of a force field on M takes the form $\mathbf{f}: M \rightarrow T^*\mathcal{Q}|_M$. Notice that the space of forces on the manifold M is the dual tangent bundle of the ambient configuration space. Inclusion of M in any other ambient space changes the space of forces on M. In this respect we consider forces on any subset with an implied ambient space - the configuration space.

Change of Coordinate System for Forces

Consider configuration space $SE(3)$. A representation $\mathbf{f}^\mathbf{x}_e$ of a force $\mathbf{f}_e$ at the identity $e \in SE(3)$ in the dual coordinate system on the cotangent space for a choice of coordinate system $\mathbf{x}$ is

$$\mathbf{f}^\mathbf{x}_e = [\,\tau^\mathbf{x}_1 \quad \tau^\mathbf{x}_2 \quad \tau^\mathbf{x}_3 \quad f^\mathbf{x}_1 \quad f^\mathbf{x}_2 \quad f^\mathbf{x}_3\,]^T. \tag{2.16}$$

Another choice of a coordinate system $\mathbf{y}$ where $\mathbf{y} = \mathbf{x} \circ \mathbf{h}^{-1}$ induces a corresponding represenation of the forces

$$\mathbf{f}^\mathbf{y}_e = Ad^{*\mathbf{x}}_\mathbf{h}(\mathbf{f}^\mathbf{x}_e) \tag{2.17}$$

where $Ad^{*\mathbf{x}}_\mathbf{h}$ is the *coadjoint* of the inverse conjugation map 2.7. Following a construction parallel to the one given in equation 2.11, a relation between the representation of forces at an arbitrary configuration $\mathbf{q}$ is

$$\mathbf{f}^\mathbf{y}_\mathbf{q} = \mathrm{Diag}(\mathbf{R}^{\mathbf{q}^\mathbf{x}}, 2 \times n)\; Ad^{*\mathbf{x}}_\mathbf{h}\; \mathrm{Diag}(\mathbf{R}^{\mathbf{q}^{-1\mathbf{x}}}, 2 \times n)\; \mathbf{f}^\mathbf{x}_\mathbf{q}. \tag{2.18}$$

Forces of Reaction without Friction

When a vertex of a body presses against an edge of another body, the force of reaction is always directed positively outwards on a normal to the edge of contact. The normal direction is a subspace and a subset of this subspace which is positively outwards is where the forces of reaction lie. For any small displacement of the vertex in contact with the edge, the force of reaction does zero work. Using this notion, we define a *Generalized Normal Force Subspace* of the forces in the configuration space. The

Generalized Forces of Reaction in the configuration space is a subset of the *Generalized Normal Force Subspace.*

We first give a constructive definition of the generalized normal force subspace. Consider a subset M of the configuration space with the inclusion map $\imath: M \rightarrow \mathcal{Q}$ and the corresponding $D\imath: TM \rightarrow T\mathcal{Q}$. The space of all displacements in M at a point $\mathbf{q}$ is the subspace $T_{\mathbf{q}}M$ of the ambient space $T_{\mathbf{q}}\mathcal{Q}$. Define the generalized normal force subspace $\mathcal{F}^{\mathbf{n}}$ at $\mathbf{q}$ to be all elements of the cotangent space $T^*_{\mathbf{q}}\mathcal{Q}$ that map all displacements in the tangent space $T_{\mathbf{q}}M$ to zero, *i.e.*,

$$\mathcal{F}^{\mathbf{n}}_{\mathbf{q}} = \{\mathbf{f_q} \in T^*\mathcal{Q} | \forall \mathbf{v_q} \in T_{\mathbf{q}}M, \mathbf{f_q}(D\imath(\mathbf{v_q})) = 0\}. \tag{2.19}$$

There is a natural isomorphism between a vector space and the second dual of this space. With this in mind, it is not hard to observe that the generalized normal force subspace is infact an intersection of the kernel of all elements in the second dual of the tangent space $T^{**}_{\mathbf{q}}M$. Let $\imath^{**}: T^{**}M \rightarrow T^{**}\mathcal{Q}$ be the second dual inclusion map induced by the inclusion of the subset M. If we denote elements of the second dual of the tangent space as $\mathbf{v}^{**}_{\mathbf{q}}$, then

$$\mathcal{F}^{\mathbf{n}}_{\mathbf{q}} = \{ \bigcap_{\mathbf{v}^{**}_{\mathbf{q}} \in T^{**}M} ker\, \imath^{**}_{\mathbf{q}}(\mathbf{v}^{**}_{\mathbf{q}})\}. \tag{2.20}$$

First, let us make some observations about this subspace. If M is free space, a subset of codimension zero of the configuration space, the inclusion of the tangent space M is an isomorphism. The kernel is the trivial element zero of the cotangent space. In free space, the reaction forces are zero. When M is a subset of codimension r, the dimension of this subspace is r. Construct the *normal generalized force bundle* $\pi = (\mathcal{F}\mathcal{Q}_{valid}, \mathcal{Q}_{valid}, \pi)$ by piecing all the local subspaces $\bigcup_{q \in \mathcal{Q}_{valid}} \mathcal{F}^{\mathbf{n}}_{\mathbf{q}}$. The set of *generalized forces of reaction* is a subset of the *normal generalized force bundle.*

When inertial forces are considered, the forces of reaction at a configuration also depend on the velocity. On this account, we construct the forces of reaction as a mapping from the tangent bundle of the configuration space to the cotangent bundle of the configuration space - in fact to the bundle of normal generalized forces. Denote the subset of generalized forces of reaction at a point $(\mathbf{q}, \mathbf{v_q}) \in T_{\mathbf{q}}\mathcal{Q}$ as $\mathcal{F}^{\mathbf{R}}(\mathbf{q}, \mathbf{v_q}) \subset \mathcal{F}^{\mathbf{n}}_{\mathbf{q}}$.

In general, the subset of forces of reaction is a multivalued map

$$\mathcal{F}^{\mathbf{R}}: T\mathcal{Q}|_{\mathcal{Q}_{valid}} \to \mathcal{F}\mathcal{Q}_{valid} \subset T^*\mathcal{Q}|_{\mathcal{Q}_{valid}}. \tag{2.21}$$

If friction is considered, the generalized forces of reaction are not a subset of the normal generalized force subspace.

Following on a recent observation about the *orthogonal* nature of the force freedom space [Mas 81, LipDuf 88], we comment that although generalized normal force is a subspace of the dual space, relative to any metric used to identify the tangent and the cotangent bundle, the generalized normal forces are the orthogonal complement of the tangent space of M in $\mathcal{Q}$ in that metric. This follows from the definition $\mathbf{f}(\mathbf{v}) = g(g^\sharp(\mathbf{f}), \mathbf{v})$ given earlier for $\mathbf{f} \in T^*M$, and $\mathbf{v} \in TM$. The generalized normal force subspace is defined in equation 2.19 by the zero set of $\mathbf{f}(\mathbf{v})$. It implies that the inner product of $g^\sharp(\mathbf{f})$ and $\mathbf{v}$ vanishes with g as metric and so they are orthogonal relative to g. To see this in coordinates, first consider the matrix $g_{ij} = g(\frac{\partial}{\partial x_k^i}, \frac{\partial}{\partial x_k^j})$ of the Riemannian metric g in local coordinates $\{U, x_k\}$ on $\mathcal{Q}$. For simplicity, use dual coordinates dx_k^i on the cotangent space, so $dx_k^i(\mathbf{q})\left(\frac{\partial}{\partial x_k^j}|_{\mathbf{q}}\right) = \delta_j^i$, the Kronecker delta, and g^{ij}, the inverse of g_{ij}, is the matrix of linear map $g^\sharp \in L(T^*M, TM)$. An element $\mathbf{f}_{\mathbf{q}} = \sum_i f_{\mathbf{q}}^i dx_k^i$, therefore, has a representation $g^\sharp(\mathbf{f}_{\mathbf{q}}) = \sum_i g^\sharp(\mathbf{f}_{\mathbf{q}})^i \frac{\partial}{\partial x_k^i} \in T_{\mathbf{q}}M$, where $g^\sharp(\mathbf{f}_{\mathbf{q}})^i = \sum_j g^{ij} f_{\mathbf{q}}^j$. Now, consider the inner product of an element $\mathbf{v}_{\mathbf{q}} \in T_{\mathbf{q}}M$ and the image $g^\sharp(\mathbf{f}_{\mathbf{q}})$ of a force element $\mathbf{f}_{\mathbf{q}} \in T_{\mathbf{q}}^*M$. In local coordinates on the tangent space and the dual coordinates on the cotangent space, it is $\sum_{ij} g_{ij} dx_k^i \otimes dx_k^j(\sum_i v_{\mathbf{q}}^i \frac{\partial}{\partial x_k^i}, \sum_{ij} g^{ij} f_{\mathbf{q}}^j \frac{\partial}{\partial x_k^i}) = \sum_{ijk} v_{\mathbf{q}}^i g_{ij} g^{jk} f_{\mathbf{q}}^k = \sum_i v_{\mathbf{q}}^i f_{\mathbf{q}}^i$.

Now, consider the generalized normal force subspace defined by the vanishing of the functional $\mathbf{f}_{\mathbf{q}}(\mathbf{v}_{\mathbf{q}})$. In dual coordinates, it reduces to $\sum_i v_{\mathbf{q}}^i f_{\mathbf{q}}^i$ whose vanishing implies that the inner product vanishes, and hence orthogonality.

2.2 Control Equations as Vector Fields

We consider control equations, uncertainty on control equations, attainable set, and forward projection on smooth sets. For simplicity, let the manifold M_i that contains a strata S_i as a closed and regular subset be simply denoted M.

2.2.1 Control Equations and Flows

An evolution of a physical system with feedback control in a continuous domain defines the passage of the state of the system as a continuous function of time. For a point $\mathbf{s}$ in the state space, it tells the time rate of change of the state $\mathbf{s}$ at any instant. Usually, the state of the system is the space of positions and velocities. In this space, a control system specifies the time derivative of the position and velocity (velocity and acceleration respectively) at all the points in the domain. A solution of the control is solution of an initial value problem of a system of ordinary differential equations. A solution represents the passage of the state as a continuous function of time. In this case, given an initial state $\mathbf{s}(0)$, the solution tells us the state $\mathbf{s}(t)$ or $\phi(t, \mathbf{s}(0))$ at a time t. Trajectories are evolutions of a state $\mathbf{s}$ as a function of time.

The trajectories on the boundaries of the configuration space obstacles correspond to one of the several (force) control schemes that allow a continuous motion to be executed in constrained space. All such force control schemes, and as a matter of fact, even plain motion control schemes, are specified as simple second-order differential equations. In this view of motion on configuration space obstacles, the role of force control is to factor out any normal component of the motion in the constrained space. The corresponding actual motion that occurs on the obstacles is the remaining tangential component. To compute the trajectories, one deduces the tangential components from the motion specification. The tangential component of the motion specifies a differential equation on the constrained surface.

First consider the bundles $\pi = (TM, M, \pi)$ and $\tau = (T(TM), TM, \tau)$, the tangent bundles of M and TM. Note that $D\pi = (T(TM), TM, D\pi)$ is also a bundle where $D\pi$ denotes the derivative of the map π. A map $X: TM \to T(TM)$ is called a *section* of the the bundle τ if $\tau \circ X$ is the identity map.

A control equation on a stratum M is a second-order equation that assigns an acceleration to every point of the tangent space $(\mathbf{q}, \mathbf{v_q}) \in TM$. Formally, consider an autonomous control equation as a vector field on TM given by a C^s map X

$$X: TM \to T(TM) \tag{2.22}$$

such that it is a section of the bundles τ and $D\pi$. If (U, ϕ, ψ) is a local tangent

bundle chart of π with $\psi: \pi^{-1}U \to \phi U \times \mathbf{R}^n$, and $(\pi^{-1}U, \psi, \eta)$ a local tangent bundle chart of τ with $\eta: \tau^{-1}(\pi^{-1}U) \to (\phi U \times \mathbf{R}^n) \times \mathbf{R}^n \times \mathbf{R}^n$, then the section X induces a local section $X': \phi U \times \mathbf{R}^n \to (\phi U \times \mathbf{R}^n) \times \mathbf{R}^n \times \mathbf{R}^n$. Let points and functions in the image of a chart map be denoted by corresponding prime letters. So, if $X'(\mathbf{q}', \mathbf{v}'_\mathbf{q}) = ((\mathbf{q}', \mathbf{v}'_\mathbf{q}), \mathbf{w}'_{(\mathbf{q}',\mathbf{v}'_\mathbf{q})}, \mathbf{a}'_{(\mathbf{q}',\mathbf{v}'_\mathbf{q})})$, then X being a section of the bundles τ and $D\pi$ implies $D\pi' \circ X' = \tau' \circ X'$ so that $D\pi' \circ X'(\mathbf{q}', \mathbf{v}'_\mathbf{q}) = (\mathbf{q}', \mathbf{w}'_{(\mathbf{q}',\mathbf{v}'_\mathbf{q})})$. Therefore, X is a second-order equation only if $\mathbf{v}'_\mathbf{q} = \mathbf{w}'_{(\mathbf{q}',\mathbf{v}'_\mathbf{q})}$. For brevity, we now use the notation $\mathbf{s} \in TM$ for the state $(\mathbf{q}, \mathbf{v}_\mathbf{q})$ together with the natural projection map $\pi(\mathbf{s}) = \mathbf{q}$.

A point $\mathbf{s}$ is called a *singular point* of the vector field if $X(\mathbf{s}) = 0$. A map X is transversal, if the linear map DX is full rank at all points $X(\mathbf{s}) = 0$. A singular point $\mathbf{s}$ is isolated if map X is transversal. If the real parts of all the eigenvalues of the linear map $DX(\mathbf{s})$ are non-zero, the singular point $\mathbf{s}$ is called *hyperbolic* and is characterized as a *source*, a *sink*, or a *saddle* when the real parts of the eigenvalues are all positive, all negative, or some positive and some negative, respectively. By a solution to the vector field, we mean maps $\phi: (-\epsilon, \epsilon) \times TM \to TM$ whose tangent vectors coincide with X, that is, a map ϕ with

$$\begin{aligned} \phi(0, \mathbf{s}) &= \mathbf{s}, \\ D_1\phi\left(\frac{\partial}{\partial t}\Big|_{(t,\mathbf{s})}\right) &= X(\phi(t, \mathbf{s})), \\ D_1(\pi\phi)\left(\frac{\partial}{\partial t}\Big|_{(t,\mathbf{s})}\right) &= \phi(t, \mathbf{s}). \end{aligned} \tag{2.23}$$

A continuous function ϕ is an indefinite integral if, and only if, it is absolutely continuous. Therefore, we consider solutions ϕ that belong to the space $AC(\mathbf{R}, TM)$ of absolutely continuous functions from an interval I to the tangent bundle TM. In general, the flow ϕ is C^0 if X is Lipschitz and the flow is C^k if X is C^k [Lang 62].

If we assume that X is a C^1 map on each stratum, then it satisfies the Lipschitz conditions locally, so basic theorems about the existence, uniqueness, and differentiability of the solutions apply. With existence and uniqueness of the flow ϕ, if there are two solutions ϕ and ϕ' defined for $t \in \mathcal{J}^u = (u - \epsilon, u + \epsilon)$ and $t \in \mathcal{J}^{u'} = (u' - \delta, u' + \delta)$ such that $\mathcal{J}^u \bigcap \mathcal{J}^{u'} \neq \emptyset$, then these solutions agree on $\mathcal{J}^u \bigcap \mathcal{J}^{u'}$. Piecing such solutions, one obtains a maximal open interval $\mathcal{J} \subset \mathcal{R}$ about every $\mathbf{s}$ called the *maximal*

interval $\mathcal{J}(\mathbf{s})$. Call the domain of the flow ϕ of a vector field X an open subset $\mathcal{D}_X$ of $\mathbf{R} \times TM$ such that $\mathcal{D}_X \supset \{0\} \times TM$, and the flow is maximal - for each $\mathbf{s} \in TM, \mathbf{R} \times \{\mathbf{s}\} \cap \mathcal{D}_X$ is open interval. Now, for each $\mathbf{s}$ consider a map $\phi_{\mathbf{s}}: \mathcal{J} \rightarrow TM$ such that $\phi_{\mathbf{s}}(t) \stackrel{def}{=} \phi_t(\mathbf{s}) \stackrel{def}{=} \phi(t, \mathbf{s})$. The map $\phi_{\mathbf{s}}$ or the whole set $\phi_{\mathbf{s}}(\mathcal{J}(\mathbf{s}))$ is called the *orbit* or the *trajectory* of the vector field passing through $\mathbf{s}$. The complete map ϕ is also called *flow* of the vector field.

We consider vector fields that depend on a parameter and the corresponding dependence of the solutions on the parameter. Let $E \subset \mathcal{R}^a$ be the domain of parameters $\mathbf{p}$. Let $X: M \times E \rightarrow TM$ be a C^s map, $s \geq 1$. For each $\mathbf{p} \in E$, the map $X_{\mathbf{p}}: M \rightarrow TM$ defined by $X_{\mathbf{p}}(\mathbf{s}) = X(\mathbf{s}, \mathbf{p})$ is a vector field of class C^s on M. The solutions $\phi(t, \mathbf{s}, \mathbf{p})$ are also C^s maps.

2.2.2 Bounded Perturbation of Vector Fields

A function X represents a perfect control equation. However, a function other than X may also determine the flow. Given a bounded perturbation of the parameters determining the motion equation, the multivalued map of equations of motion is characterized by a proposition from Aubin and Cellina [AubCel 84]:

Proposition 2.2 *Assume that the base space M, the tangent space TM, and the space of parameters $\mathcal{U}$ are manifolds, and the map $X: M \times \mathcal{U} \rightarrow TM$ is a continuous and proper section, the space $\mathcal{U}$ is a compact subset, and*

$$\forall \mathbf{u} \in \mathcal{U}, \mathbf{x} \mapsto X(\mathbf{x}, \mathbf{u}) \tag{2.24}$$

is continuous, then the set valued map

$$F^0(\mathbf{x}) = \{X(\mathbf{x}, \mathbf{u}) | \mathbf{u} \in \mathcal{U}\} \tag{2.25}$$

is continuous and compact.

Consider that the external forces, such as in equation 1.8, are known to be within a bound. Pick an external force that is an element in this bound, and determine the corresponding possible motions. We take the union of all such motions for each force

element in the bounded region. It is possible in this way to also relax the assumption of precise inertial parameters and construct a union of all possible motions when these parameters are changed within a prescribed bound. The multivalued map thus constructed bounds the possible deviations from X.

2.2.3 Attainable Sets and Forward Projection

An attainable set of a point are states reachable at a given instant. Forward projection set of a point are all such reachable states.

For a vector field on a constrained subset like the boundary of a configuration space obstacle, any actual physical motion is realizable if an appropriate *forces of constraint* is available. The simplest example is that of a particle mass moving in a circular orbit, a constrained subset of the Euclidean plane. The external forces acting on this particle must be equal to the centrifugal force if the particle is to stay in the circular orbit. If external forces are given as a function of configuration, and velocity, the subset of the constrained space, where appropriate forces of constraint are met by the physical nature of the constraint, is the subset where motion occurs on this stratum. The attainable set and the forward projection on each stratum is thus limited to this subset. Let V denote such a subset of TM.

For $\mathbf{s}$ in smooth subsets $V \subset TM$, consider the *Cauchy Problem* of finding all absolutely continuous functions $\phi: \mathbf{R} \times V \to TM$ such that

$$\begin{aligned} \phi(0,\mathbf{s}) &= \mathbf{s}, \\ D_1\phi\left(\frac{\partial}{\partial t}\Big|_{(t,\mathbf{s})}\right) &\in F^0(\phi(t,\mathbf{s})), \quad \textit{almost everywhere}, \\ D_1(\pi\phi)\left(\frac{\partial}{\partial t}\Big|_{(t,\mathbf{s})}\right) &= \phi(t,\mathbf{s}). \end{aligned} \tag{2.26}$$

This is a problem in differential inclusion and a direct generalization of ordinary differential equations such as equation 2.23. Let I be the interval $[0,T]$ for $T > 0 \in \mathbf{R}$. Consider the space of all absolutely continuous maps $AC(I,TM)$. For any point $\mathbf{s} \in V$, let $\Phi(\mathbf{s}) \subset AC(\mathbf{R},TM)$ denote the set of solutions to the Cauchy problem 2.26. Denote $\mathcal{J}_{\phi\mathbf{s}}$ as the interval of time for which a solution ϕ is defined over V.

Definition 2.2 *Define the attainable set of an initial set $\mathcal{I}$ at a time $t > 0$ as*

$$\mathcal{A}(t, \mathcal{I}) = \{\mathrm{s}' \in V | \mathrm{s} \in \mathcal{I}, \phi_{\mathrm{s}} \in \Phi(\mathrm{s}),\, t \in \mathcal{J}_{\phi_{\mathrm{s}}},\, \mathrm{s}' = \phi(t, \mathrm{s})\}. \tag{2.27}$$

Definition 2.3 *Define forward projection of an initial set $\mathcal{I}$ as*

$$\mathrm{FP}(\mathcal{I}) = \{\mathrm{s}' \in V | \mathrm{s} \in \mathcal{I}, \phi_{\mathrm{s}} \in \Phi(\mathrm{s}), t \in \mathcal{J}_{\phi_{\mathrm{s}}}, \mathrm{s}' = \phi(t, \mathrm{s})\}. \tag{2.28}$$

An attainable set is also a forward projection - albeit subsets given as a function of time. However, we restrict the term forward projection to the time independent subset defined here. Forward projection is also known as an *Integral Funnel*, or a *Reachable Set.*

If a model of a rigid body collision is assumed, it is possible to extend the definitions of attainable set, and the forward projection from each such smooth set V to the whole space CQ_{valid}, the piecewise smooth tangent cone bundle of the valid configuration space.

2.3 Fine Motion Planning Problem

2.3.1 Sensing Uncertainty

A robot is equipped with sensors, for example, *position sensor* or *force sensor.* Uncertainty in any sensing is defined by a *multivalued map.* A *multivalued map* η assigns to each point of the domain an element in the set of subsets of the range. A *Measurement* or a *Sensing Query* is a single valued selection from the image of the multivalued sensing uncertainty map. When we say "for all sensing ...," we mean all sensing queries consistent with the multivalued sensing uncertainty map - informally, all measurements consistent with the sensor model.

Position sensing uncertainty is a multivalued map from the configuration space to itself. Denote the uncertainty associated with a position sensor as a multivalued map $\eta_{\mathcal{Q}}\colon \mathcal{Q}_{valid} \rightarrow \mathcal{Q}$. For example, $\eta_{\mathcal{Q}}(\mathbf{q}) = \{\mathbf{q}^* \in \mathcal{Q} | \mathbf{q}^* \in \Sigma(\mathbf{q}, \rho)\}$, where $\Sigma(\mathbf{q}, \rho)$ is a ball of radius ρ with center at $\mathbf{q}$. We use the standard convention that any measured (sensed) quantity is denoted by letters with * as superscript (Elements

in the cotangent space are likely to be confused with measured elements.). The actual (real) quantities are denoted with plain letters. So, uncertainty in the position sensing defined in the example is a multivalued map that assigns to each actual configuration $\mathbf{q}$ a subset of the configuration space. Each element $\mathbf{q}^* \in \eta_{\mathcal{Q}}(\mathbf{q})$ is a possible measurement that a sensor at $\mathbf{q}$ is likely to return. Note that the domain of the map $\eta_{\mathcal{Q}}$ are places where the sensor can be placed which is $\mathcal{Q}_{valid}$, but sensed values can be inside the C-obstacles, so the range is $\mathcal{Q}$ (although, the range of the map $\eta_{\mathcal{Q}}$ is possibly a proper subset of $\mathcal{Q}$).

Velocity sensors are rare; but practically every robot controller uses position derivative data as an estimate of the velocity. Consider a model of the velocity sensing uncertainty as a bundle multivalued map $\eta_{\mathbf{V}}: C\mathcal{Q}_{valid} \rightarrow T\mathcal{Q}|_{\mathcal{Q}_{valid}}: \mathbf{s} \overset{\eta_{\mathbf{V}}}{\mapsto} \eta_{\mathbf{V}}(\mathbf{s}) \subset T_{\pi(\mathbf{s})}\mathcal{Q}$. For example, $\eta_{\mathbf{V}}(\mathbf{q}, \mathbf{v_q}) = \{\mathbf{v}^*_{\mathbf{q}} \in T_{\mathbf{q}}\mathcal{Q} | \mathbf{v}^*_{\mathbf{q}} \in \Sigma_{\mathbf{q}}(\mathbf{v_q}, \rho)\}$, where $\Sigma_{\mathbf{q}}(\mathbf{v_q}, \rho)$ is a ball of radius ρ and center $\mathbf{v_q}$. Note that if a point $\mathbf{q}$ lies on a proper subset of the valid configuration space, the velocity sensing query need not be restricted to its tangent space.

A force sensor is a multivalued map from the dual of the tangent bundle of the configuration space to itself. Its model is a bundle multivalued map $\eta_{\mathbf{f}}: T^*\mathcal{Q}|_{\mathcal{Q}_{valid}} \rightarrow T^*\mathcal{Q}|_{\mathcal{Q}_{valid}}$. For example, $\eta_{\mathbf{f}}(\mathbf{q}, \mathbf{f_q}) = \{\mathbf{f}^*_{\mathbf{q}} \in T^*_{\mathbf{q}}\mathcal{Q} | \mathit{angle}(\mathbf{f}^*_{\mathbf{q}}, \mathbf{f_q}) < \epsilon_\theta, \mathit{magnitude}|\mathbf{f}^*_{\mathbf{q}} - \mathbf{f_q}| < \epsilon_f\}$, where an appropriate norm and inner product is defined on the cotangent space, models a force sensor which to a given force $\mathbf{f_q}$, assigns a subset of forces that differ in magnitude and angle from $\mathbf{f_q}$ by a prescribed amount. Each element of $\eta_{\mathbf{f}}(\mathbf{q}, \mathbf{f_q})$ is a possible force that the force sensor at a point $\mathbf{q}$ subjected to a force $\mathbf{f_q}$ is likely to return upon query.

Let us denote the space of measurements by $\mathbf{E}$. For example, with position, velocity, and force sensor, it is a vector bundle over the valid configuration space

$$\sigma = \{\mathbf{E} = \bigcup_{\mathbf{q} \in \mathcal{Q}_{valid}} T_{\mathbf{q}}\mathcal{Q} \times T^*_{\mathbf{q}}\mathcal{Q}, \mathcal{Q}_{valid}, \sigma\}. \tag{2.29}$$

The fibers of the bundle, in this case, are direct sums of the tangent and the cotangent spaces of the valid configuration space. In general, the sensing uncertainty can be modelled as follows:

Definition 2.4 *The sensing uncertainty is modelled as a multivalued map η from the*

tangent cone of the valid configuration space to $\mathbf{E}$*, the space of measurements, i.e.,*

$$\eta: CQ_{valid} \rightarrow \mathbf{E}.$$

With position, velocity and force sensor, map $\eta \equiv \{\eta_Q, \eta_\mathbf{v}, \eta_\mathbf{f}\}$.

Given sensing uncertainty model η, it is now possible to construct a multivalued map $\mathcal{K}^*$ which is consistent with a given state and the geometric model. The necessity of geometric model is illustrated by the force sensor. First, consider a state $\mathbf{s} \equiv (\mathbf{q}, \mathbf{v_q})$. It is trivial to combine the position and velocity sensing uncertainty maps and construct the set of sensing queries consistent with a state s. This map is $\eta_\mathbf{s}: CQ_{valid} \rightarrow TQ|_{Q_{valid}}: \mathbf{s} \stackrel{\eta_\mathbf{s}}{\mapsto} (\eta_Q(\pi(\mathbf{s})), \eta_\mathbf{v}(\mathbf{s}))$. Now, if a sensing query also contains force measurement, it is necessary to know the possible forces that the force sensor is likely to be subjected to in order to construct possible measurements. Let us recall that the set of possible reaction forces, as a function of the state, is constructed in equation 2.21. Then, the *consistent measurement map* $\mathcal{K}^*$ is defined as

$$\mathcal{K}^*: \mathbf{C}Q_{valid} \rightarrow \mathbf{E}: \mathbf{s} \stackrel{\mathcal{K}^*}{\mapsto} \bigcup_{\mathbf{f}_{\pi(\mathbf{s})} \in \mathcal{F}^\mathbf{R}(\mathbf{s})} (\eta_\mathbf{s}(\mathbf{s}), \eta_\mathbf{f}(\pi(\mathbf{s}), \mathbf{f}_{\pi(\mathbf{s})})). \tag{2.30}$$

Let us denote a measurement by $\mathbf{m}^*$. A measurement is a single-valued selection of the space of consistent measurements, *i.e.,* $\mathbf{m}^*: CQ_{valid} \rightarrow \mathbf{E}: \mathbf{s} \mapsto \mathbf{m}^*(\mathbf{s}) \in \mathcal{K}^*(\mathbf{s})$. With position, velocity, and force sensor, $\mathbf{m}^*(\mathbf{s}) \equiv (\mathbf{q}^*(\pi(\mathbf{s})), \mathbf{v}^*_{\pi(\mathbf{s})}(\mathbf{s}), \mathbf{f}^*_{\pi(\mathbf{s})}(\mathbf{s}))$. In this respect, $\mathbf{m}^*$ is a section of the measurement bundle. We do not assume that this section is continuous. Consider now sensor measurements on a trajectory in the phase space. Let $\mathbf{m}^*_{\phi_\mathbf{s}}$ denote measurements on a trajectory $\phi_\mathbf{s}$ such that $\mathbf{m}^*_{\phi_\mathbf{s}}: \{0\} \cup \mathbf{R}^+ \rightarrow E: t \mapsto \mathbf{m}^*(\phi_\mathbf{s}(t))$. A history of measurements on a trajectory $\phi_\mathbf{s}$ is the set of measurements from the start of time to time t, defined as

$$\mathbf{m}^*_{\phi_\mathbf{s}}[0,t] = \{\mathbf{m}^*(\phi_\mathbf{s}(t')) \in \mathcal{K}^*(\phi_\mathbf{s}(t'))|t' \in [0,t]\} \tag{2.31}$$

Although it is possible to write a history of measurements as $\mathbf{m}^*[0,t]$ without any reference to a trajectory, it is misleading since the set of states at all times in the interval including at the start of time $t = 0$ is not apparent. A history of measurements, $\mathbf{m}^*_{\phi_\mathbf{s}}$, is an *observed trajectory.*

2.3.2 Motion Command and Termination Condition

A *Motion Command* **M** is denoted by a pair (**CS**,**TC**) of the *Control Statement* **CS** and the *Termination Condition* **TC**. A control statement determines the time history of the state as a multivalued map, *i.e.,* either an attainable subset of position and velocity as a function of time or a subset of all attainable positions and velocities.

In its simplest form, a *Termination Condition* is a boolean function whose state is determined by current and past sensing queries. Typically, we would like to interpret a sensing query to determine all possible states of the system consistent with the query. If these interpretations are subset of some desired goal, termination condition signals the end of the control statement. Define a *termination predicate* to be a compilation of the boolean-valued termination condition. In particular, it compiles the planning time inputs such as sensing uncertainty, control uncertainty, geometric world model, the termination condition itself, and a description of the goal set $\mathcal{G} = \{\mathcal{G}_\alpha\}$. The interpretation of sensing queries are compiled to a predicate that evaluates to true iff the set of interpretation is a subset of the goal. Erdmann [Erd 86] classifies such termination conditions into several forms, namely, *standard termination predicate*, *termination predicate with state*, *termination predicate with no sense of time*, and *termination predicate without history or time.* All such predicates can be expressed as

$$\mathbf{tp}: 2^{\mathbf{E}} \rightarrow \{true, false\}. \tag{2.32}$$

In general, a termination condition can be a function into the space of motion commands **M** from the set of current and past sensing queries. In this general sense, a termination predicate not only signals the end of execution of the current control statement but also determines the subgoal achieved to determine the next motion command to be executed. See Mason [Mas 84] for more details.

2.3.3 One-Step Preimage

The problem of *One-Step Fine Motion Plan* for a given goal $\mathcal{G} = \{\mathcal{G}_\alpha\}$, and a set of initial configurations $\mathcal{I}$, is to find a motion command $\mathbf{M} = (\mathbf{CS}, \mathbf{TC})$ such that: given planning inputs in equation 1.1, if the robot is known to start in the region $\mathcal{I}$, then

executing $\mathbf{M}$ is guaranteed to take the robot inside $\mathcal{G}$ (goal reachability) and stop in it (goal recognizability) [LMT 84, Mas 84, Erd 84, Lat 88]. This simple, but rather elusive, statement implies that without knowing the evolution of sensing queries or control trajectory a-priori, the robot reaches the goal and the termination condition is guaranteed to signal success when the robot is inside the goal.

Consider the two clauses in the definition of predicate *Achieve* in equation 1.2. For every trajectory $\phi_{\mathbf{s}} \in \Phi(\mathbf{s})$ condition (i) asserts that either in finite time the system *certainly stops* in the goal or otherwise the asymptotic point must be part of the goal. For a perpetually false termination predicate, and a phase flow of a smooth field with singularity, there are trajectories that never reach the singular point in any finite time - the singular point is part of the goal (see example 2.1(c)). The second condition, (ii), affirms that if ever the termination predicate can signal success before certainly coming to stop, it does so only when the system is in the goal.

Before giving some simple one-dimensional examples, we reassert essential definitions and properties [LMT 84, Mas 84, Erd 84, Lat 88]:

Definition 2.5 *A One-Step Preimage of a goal* $\mathcal{G} = \{\mathcal{G}_\alpha\}$ *for a motion command* $\mathbf{M} = (\mathbf{CS}, \mathbf{TC})$ *is a subset* $\mathcal{P}_{\mathbf{M}}(\mathcal{G}) \subset V$ *such that* $\texttt{Achieve}(\mathcal{G}, \mathbf{M}, \mathcal{P}_{\mathbf{M}}(\mathcal{G}))$.

It follows from the definition of preimage that

- A subset of a preimage is also a preimage for the same motion command $\mathbf{M}$, and
- A union of two preimages is also a preimage for the same motion command $\mathbf{M}$.

Using these properties, the *maximal preimage* for a given motion command $\mathbf{M}$ is defined as

Definition 2.6 *The subset of* CQ_{valid}

$$\mathcal{P}_{\mathbf{M}}^{MAX}(\mathcal{G}) = \{\mathbf{s} \in V | \texttt{Achieve}(\mathcal{G}, \mathbf{M}, \{\mathbf{s}\})\}$$

is the maximal preimage of goal $\mathcal{G}$ *for the motion command* $\mathbf{M}$.

A union of maximal preimages may not be a maximal preimage, though it is always a preimage, *i.e.*,

$$\mathcal{P}_{\mathbf{M}}^{MAX}(\mathcal{G}_1) \cup \mathcal{P}_{\mathbf{M}}^{MAX}(\mathcal{G}_2) \subseteq \mathcal{P}_{\mathbf{M}}^{MAX}(\mathcal{G}_1 \cup \mathcal{G}_2).$$

Example 2.1 We give some simple examples of preimages in one-dimensional world. Consider $\mathcal{Q} = \mathcal{Q}_{valid} = \mathbf{R}$ as the configuration space that has no obstacles. Only position sensing is available so that $\eta_{\mathcal{Q}}(q) = \{q^* \mid |q - q^*| \leq \rho_q\}; \rho_q > 0$ is the model of the position sensor (see Figure 2.1).

a. The control is modelled by $\mathbf{CS}_0 \equiv \{\phi_q(t) = q + a\,t\}$ where $a \in \mathbf{R}, a > 0$ is a scalar constant denoting the velocity. Consider as Goal, $\mathcal{G}_0 = AB$, where AB is a closed line segment of length $2\rho_q$. If we consider as termination condition $\mathbf{TC}_0 \equiv \{(A + \rho_q \leq q^* \leq A + 3\rho_q)\}$, then the region $(-\infty, B]$ is the preimage $\mathcal{P}_{(\mathbf{CS}_0,\mathbf{TC}_0)}(\mathcal{G}_0)$.

b. The control is modelled by $\mathbf{CS}_0$. Consider as Goal $\mathcal{G}_1 = AB \cup CD$ where subgoals AB and CD are closed line segments defined by points as $B = A + 2\rho_q$, $D = C + 2\rho_q$, and $C > B$. The goal consists of two line segments AB and CD, AB strictly to the left of CD. If we consider as termination condition $\mathbf{TC}_1 \equiv \{(q^* = A + \rho_q) \vee (C + \rho_q \leq q^* \leq C + 3\rho_q)\}$, then the region $(-\infty, D]$ is the preimage $\mathcal{P}_{(\mathbf{CS}_0,\mathbf{TC}_1)}(\mathcal{G}_1)$.

c. The control is perfect and modelled by $\mathbf{CS}_1 \equiv \{\phi_q(t) = q\exp(-t)\}$. Consider as goal $\mathcal{G}_2 = \{0\}$, the point at the origin. If the termination condition $\mathbf{TC}_2 \equiv \{false\}$ is the perpetually false predicate then the whole of the real line $\mathbf{R}$ is the preimage $\mathcal{P}_{(\mathbf{CS}_1,\mathbf{TC}_2)}(\mathcal{G}_2)$.

d. The control is modelled by $\mathbf{CS}_0$. Consider as goal $\mathcal{G}_3 = AB$, the closed line segment defined by points $B = A + 2\rho_q - \epsilon; 0 < \epsilon < 2\rho_q$. It is of length slightly shorter than the diameter of the position sensing uncertainty. Consider as termination condition $\mathbf{TC}_3 \equiv \{A + \rho_q \leq q^* \leq A + 3\rho_q\}$. The preimage is the null region *i.e.* $\mathcal{P}_{(\mathbf{CS}_0,\mathbf{TC}_3)}(\mathcal{G}_3) = \emptyset$.

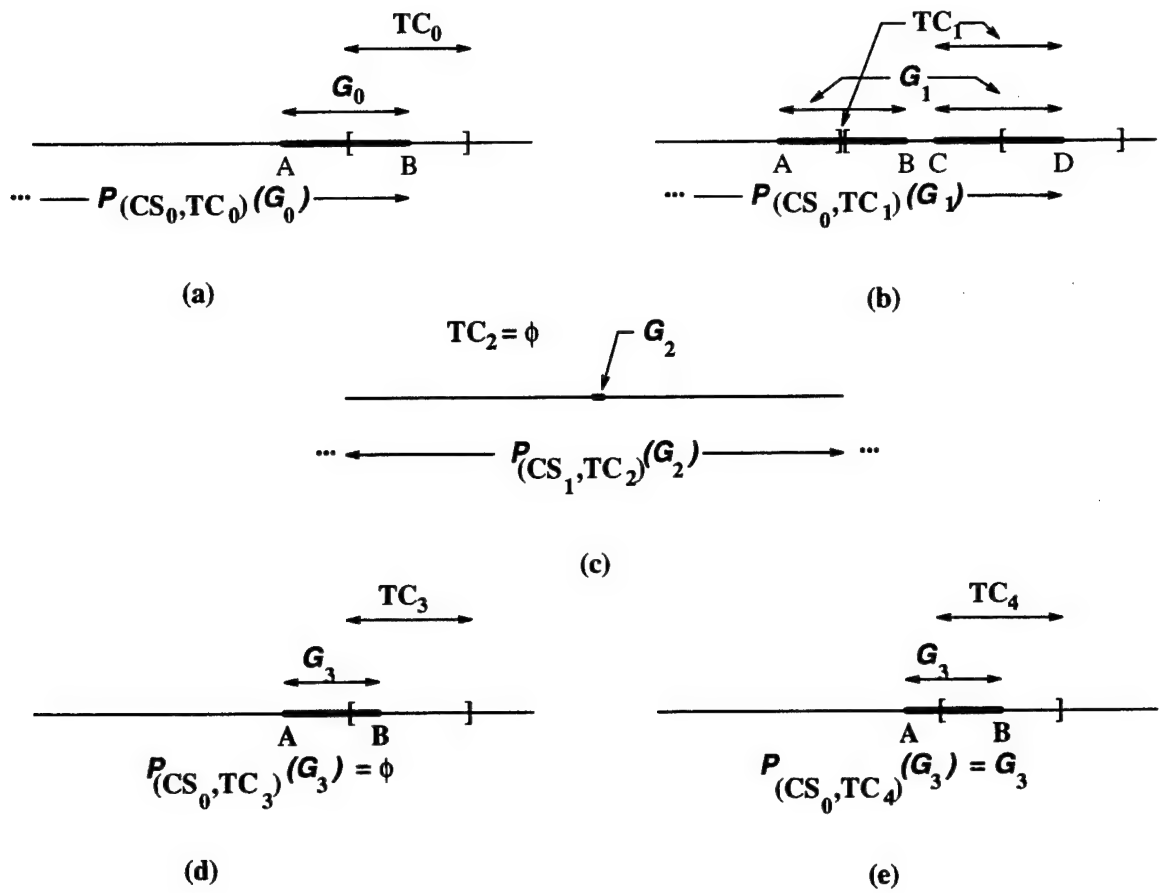


Figure 2.1: Simple One-Dimensional Examples of Preimages

e. The control is modelled by $\mathbf{CS}_0$. The goal is $\mathcal{G}_3$, same as in the previous example. Consider as termination condition $\mathbf{TC}_4 \equiv \{B - \rho_q \leq q^* \leq B + \rho_q\}$. The preimage is the region equal to the goal itself *i.e.* $\mathcal{P}_{(\mathbf{CS}_0,\mathbf{TC}_4)}(\mathcal{G}_3) = \mathcal{G}_3$.

◇

2.4 Constructing Some Preimages

In this section, we present some tools for constructing examples of preimages using *instantaneous sensing queries* for termination. Our preimage definition is like a program verifier. It accepts as input the goal $\mathcal{G}$, the motion command $\mathbf{M} = (\mathbf{CS}, \mathbf{TC})$,

and the preimage region $\mathcal{P}_{\mathbf{M}}(\mathcal{G})$ and asserts its correctness with regard to *Goal Reachability* and *Goal Recognizability*. In contrast, a planner must construct the motion command $\mathbf{M}$ and the corresponding preimage region $\mathcal{P}_{\mathbf{M}}(\mathcal{G})$ for a given goal $\mathcal{G}$. In the following, we consider constructing maximal preimages. We follow the program of Erdmann [Erd 86] in separating the reachability and recognizability. It will become apparent that reachability and recognizability depend subtly on each other. The definitions, therefore, involve recursive functions. The program, thus, is to first hypothesize a control statement. We then construct a recognizable set of the goal called *kernel* [Erd 84, Lat 88]. *Kernel* depends on both the control statement and the termination predicate. With a hypothesized control statement, a termination predicate can be constructed if possible interpretations of the sensing query are known. We present *Sensing Interpretations* which are *mutually consistent*, consistent with the *initial state* and consistent with the *final state.* An interpretation that uses the knowledge of the preimage is an interpretation with the *initial state.* An interpretation that in addition also uses the knowledge of the termination predicate is an interpretation with the *final state.*

With the hypothesized control statement and a recursive definition of the termination predicate, we give a definition of the *kernel.* The preimage is a backprojection of the *kernel.*

2.4.1 Consistent Interpretation and Preimage without State

Consistent Interpretations

Given sensing queries, we would like to ascertain the set of states that are consistent with the queries. In the presence of several sensors, a consistent set of interpretations is likely to reduce the uncertainty in a state. For example consider two sensor queries - a position sensing query and a velocity sensor query. We consider first the position sensing query by itself; and build a subset of all possible configurations that are likely interpretations of the position sensing query. This interpretation set must be a subset of the valid configuration space. Effectively, we used the model of the geometric world to restrict the interpretation to the valid configuration space. Add to these

interpretations the set of interpretations of the velocity sensing query. This set is a consistent set of interpretations of the position and velocity sensing query - mutually consistent with each other and with the model of the world geometry, albeit only instantaneously in this construction.

Consider in general, an interpretation of any measurement $\mathbf{m}^*$ to be a subset of the configuration and velocity space. Denote by $\mathcal{K}$ the multivalued map of consistent interpretations. This map takes elements $\mathbf{m}^*$ from the space of measurements to an element of the set of subsets of the tangent bundle, *i.e.*,

$$\mathcal{K}: \mathbf{E} \rightarrow \mathbf{C}\mathcal{Q}_{valid}$$

As an example, with position, velocity, and force sensor, the interpretation map $\mathcal{K}$ is

$$\begin{aligned}\mathcal{K}(\mathbf{m}^*) \equiv \mathcal{K}(\mathbf{q}^*, \mathbf{v}_\mathbf{q}^*, \mathbf{f}_\mathbf{q}^*) = \{(\mathbf{q}, \mathbf{v}_\mathbf{q}) \in \mathbf{C}\mathcal{Q}_{valid} | \forall \mathbf{q} \in \mathcal{Q}_{valid}, \mathbf{q}^* \in \eta_\mathcal{Q}(\mathbf{q}),\\ \exists \mathbf{v}_\mathbf{q} \in \mathbf{C}_\mathbf{q}\mathcal{Q}_{valid}, \mathbf{v}_\mathbf{q}^* \in \eta_\mathbf{v}(\mathbf{q}, \mathbf{v}_\mathbf{q}),\\ \exists \mathbf{f}_\mathbf{q} \in \mathcal{F}^\mathbf{R}(\mathbf{q}, \mathbf{v}_\mathbf{q}), \mathbf{f}_\mathbf{q}^* \in \eta_\mathbf{f}(\mathbf{q}, \mathbf{f}_\mathbf{q})\} \qquad (2.33)\end{aligned}$$

Preimage without State

Two states $\mathbf{s}_1$ and $\mathbf{s}_2$ in $C\mathcal{Q}_{valid}$ are said to be **distinguishable** iff:

$$\{\mathbf{m}^* \ / \ \mathbf{s}_1, \mathbf{s}_2 \in \mathcal{K}(\mathbf{m}^*)\} = \emptyset.$$

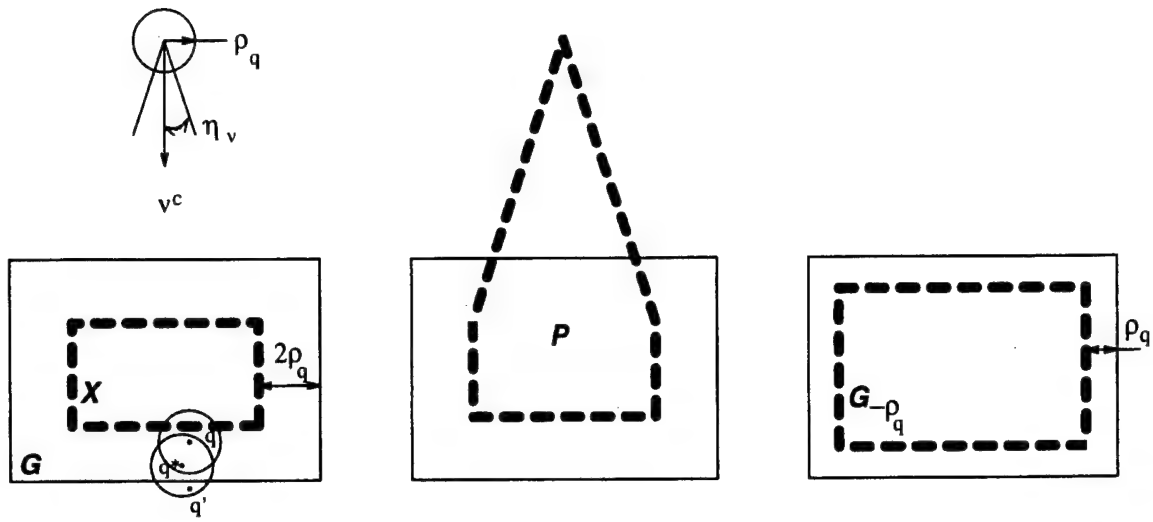
In other words, if two states $\mathbf{s}_1$ and $\mathbf{s}_2$ are distinguishable, it is guaranteed at planning time that during a motion there is no instant when the sensory data are consistent with both $\mathbf{s}_1$ and $\mathbf{s}_2$. The subset of $\mathcal{G}$ defined as:

$$\chi(\mathcal{G}) \stackrel{\text{def}}{=} \{\mathbf{s} \in \mathcal{G} \ / \ \forall \mathbf{s}' \in C\mathcal{Q}_{valid} \backslash \mathcal{G}, \mathbf{s} \text{ and } \mathbf{s}' \text{ are distinguishable}\}$$

is called the **kernel** of $\mathcal{G}$. If the robot is in $\chi(\mathcal{G})$, the only states in $C\mathcal{Q}_{valid}$ that are consistent with the current sensory data are all in the goal.

Let $\mathcal{B}_{\mathbf{CS}}(\chi(\mathcal{G}))$ be the backprojection of $\chi(\mathcal{G})$ for some control statement **CS**. It is a preimage of $\mathcal{G}$ for the motion command $(\mathbf{CS}, \mathbf{TC})$ with:

$$\mathbf{TC} \quad \equiv \quad \mathcal{K}(\mathbf{m}^*) \subseteq \mathcal{G}.$$



(a) (b) (c)

Figure 2.2: Termination without State

Indeed, by definition of $\mathcal{B}_{\mathbf{CS}}$, a motion starting from within $\mathcal{B}_{\mathbf{CS}}(\chi(\mathcal{G}))$ and commanded according to **CS** is guaranteed to reach $\chi(\mathcal{G})$ (if it is not terminated before). By definition of $\chi(\mathcal{G})$, it is guaranteed that the termination condition will become `true` at some instant during the motion. The condition will certainly become `true` when $\chi(\mathcal{G})$ is attained, but it may become `true` before. When the termination condition becomes `true`, the definition of $\mathcal{K}$ guarantees that the robot is in the goal, even if $\chi(\mathcal{G})$ has not been achieved yet. The above termination condition is called a termination condition **without state** [Lat 88].

Example 2.2 The robot is a point in a two-dimensional configuration space $\mathcal{Q} = \mathbf{R}^2$, and there are no obstacles. Hence, $\mathcal{Q} = \mathcal{Q}_{valid}$. The robot is commanded to move along a direction $\vec{\mathbf{v}}^c$ (we write $\mathbf{CS} = \vec{\mathbf{v}}^c$). Due to errors in control, it may move along any trajectory whose tangent remains contained in a cone of angle η_v whose axis points along $\vec{\mathbf{v}}^c$. The robot can only sense its current configuration and we denote the sensed configuration by $\mathbf{q}^*$. Hence, $\mathbf{m}^* = \mathbf{q}^*$. Due to errors in sensing, the actual configuration $\mathbf{q}$ of the robot may be anywhere in a disc of radius ρ_q centered at $\mathbf{q}^*$. Both η_v and ρ_q are given constants specifying the uncertainty in control and in sensing. The goal $\mathcal{G}$ is a rectangle with all its sides longer than $4\rho_q$. The vector $\vec{\mathbf{v}}^c$

is perpendicular to an edge of $\mathcal{G}$ as shown in Figure 2.2(a).

The kernel $\chi(\mathcal{G})$ of $\mathcal{G}$ is obtained by shrinking $\mathcal{G}$ by $2\rho_q$ as shown in Figure 2.2(a). Indeed, an actual configuration $\mathbf{q}$ of the robot may be sensed as any configuration $\mathbf{q}^*$ inside the disc of radius ρ_q centered at $\mathbf{q}$. The set of configurations $\mathbf{q}'$ consistent with any $\mathbf{q}^*$ in this disc forms another disc of radius $2\rho_q$ centered at $\mathbf{q}$. Hence, if $\mathbf{q}$ is in $\chi(\mathcal{G})$, this second disc lies entirely in $\mathcal{G}$, while if $\mathbf{q}$ is outside $\chi(\mathcal{G})$ (even very slightly) the second disc lies partly outside $\mathcal{G}$. The backprojection $\mathcal{P} = \mathcal{B}_{\vec{v}^c}(\chi(\mathcal{G}))$ is shown in Figure 2.2(b). It is the union of $\chi(\mathcal{G})$ and a truncated cone of angle $2\eta_v$ fitting $\chi(\mathcal{G})$. The rectangle $\mathcal{G}_{-\rho_q}$ displayed in Figure 2.2(c) depicts the set of sensed configuration $\mathbf{q}^*$ for which the termination condition evaluates to `true`. The condition $\mathbf{q}^* \in \mathcal{G}_{-\rho_q}$ can be seen as a compiled form of the termination condition $\mathcal{K}(\mathbf{m}^*) \subseteq \mathcal{G}$.$\diamond$

2.4.2 Interpretation and Preimage with Initial State

Interpretations with Initial State

A bounded model of control uncertainty on a control statement **CS** and a prior knowledge of a subset of possible initial states allows us to predict all possible future states. If the possible initial subset of states is denoted $\mathcal{I} \subset \mathbf{C}\mathcal{Q}_{valid}$, define the *initial state* as $\mathrm{FP}_{\mathbf{CS}}(\mathcal{I})$ as in Definition 2.3. We build an interpretation that is consistent not only mutually with other sensors, but also with the initial state. Given a sensing query vector $\mathbf{m}^*$, consider the set of interpretations $\mathcal{K}_{\mathbf{CS},\mathcal{I}}(\mathbf{m}^*)$

$$\mathcal{K}_{\mathbf{CS},\mathcal{I}}(\mathbf{m}^*) = \mathcal{K}(\mathbf{m}^*) \bigcap \mathrm{FP}_{\mathbf{CS}}(\mathcal{I}) \tag{2.34}$$

Preimage with Initial State

Two configurations $\mathbf{s}_1$ and $\mathbf{s}_2$ in $C\mathcal{Q}_{valid}$ are said to be **CS-$\mathcal{P}$-distinguishable** iff:

$$\{\mathbf{m}^* \;/\; \mathbf{s}_1, \mathbf{s}_2 \in \mathcal{K}_{\mathbf{CS},\mathcal{P}}(\mathbf{m}^*)\} = \emptyset.$$

The **CS-$\mathcal{P}$-kernel** of $\mathcal{G}$ is defined as:

$$\chi_{\mathbf{CS},\mathcal{P}}(\mathcal{G}) \stackrel{\text{def}}{=} \{\mathbf{s} \in \mathcal{G} \;/\; \forall \mathbf{s}' \in C\mathcal{Q}_{valid} \backslash \mathcal{G} : \mathbf{s} \text{ and } \mathbf{s}' \text{ are } \mathbf{CS} - \mathcal{P} - \text{distinguishable}\}.$$

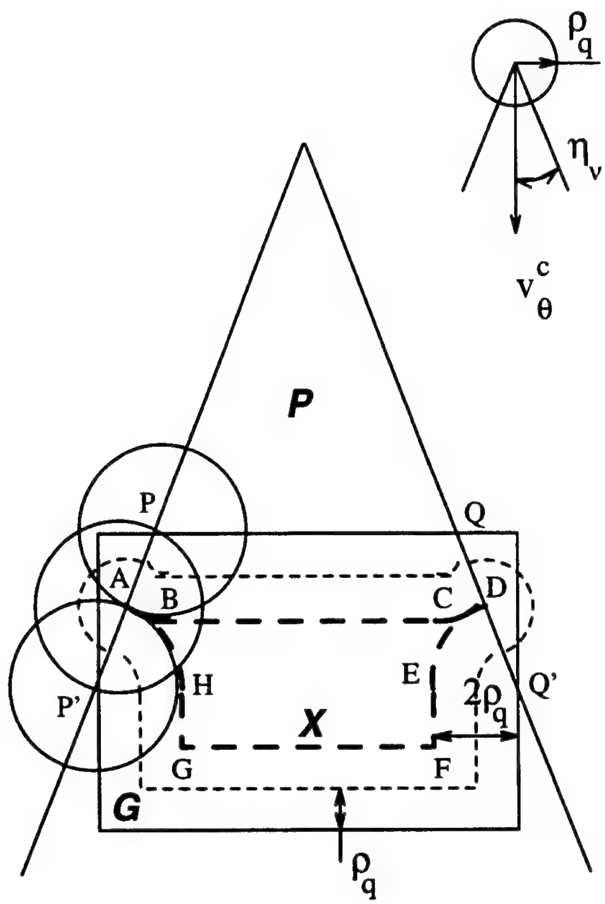


Figure 2.3: Termination with Initial State

$\mathcal{P}$ is a preimage of $\mathcal{G}$ for the motion command $(\mathbf{CS}, \mathbf{TC})$, with:

$$\mathbf{TC} \quad \equiv \quad \mathcal{K}_{\mathbf{CS},\mathcal{P}}(\mathbf{m}^*) \subseteq \mathcal{G},$$

iff it is equal to the backprojection of the $\mathbf{CS}$-$\mathcal{P}$-kernel for the control statement $\mathbf{CS}$, i.e. if $\mathcal{P} = \mathcal{B}_{\mathbf{CS}}(\chi_{\mathbf{CS},\mathcal{P}}(\mathcal{G}))$. (An equivalent recursive equation was previously established in [LMT 84].) The above condition **TC** is called a termination condition with **initial state** [SL 91, LLS 91].

Example 2.3 Consider the example of Figure 2.2 again. The region whose contour is labeled $ABCDEFGH$ in Figure 2.3 is a generalized polygon constructed as follows. The straight edge BC is at a distance $2\rho_q$ from the top horizontal edge of $\mathcal{G}$. The

circular edges AB and CD are circular arcs of radius $2\rho_q$ centered at P and Q, respectively. P (resp. Q) are selected in the top horizontal edge of $\mathcal{G}$ such that the intersection of $\mathcal{G}$ and a line passing through P (resp. Q) and parallel to the left (resp. right) side of the control uncertainty cone is a segment PP' (resp. QQ') of length $4\rho_q$. The circular edges AH and DE are circular arcs of radius $2\rho_q$ with centers at P' and Q', respectively. The straight edge GF is at distance $2\rho_q$ from the bottom edge of $\mathcal{G}$. The straight edges HG and EF are at distance $2\rho_q$ of the left and right edges of $\mathcal{G}$, respectively.

It is rather easy to verify that the region thus outlined is the kernel $\chi_{\mathbf{CS},\mathcal{P}}(\mathcal{G})$ with $\mathcal{P} = \mathcal{B}_{\mathbf{CS}}(\chi_{\mathbf{CS},\mathcal{P}}(\mathcal{G}))$. In particular, assume that at some instant during the motion the actual configuration is the extreme point marked A in the figure. All the possible sensed configurations at this instant lie in the disc of radius ρ_q centered at A. If the forward projection is not taken into account, the set of all the interpretations of all these measurements is the disc of radius $2\rho_q$ centered at A. The intersection of this disc with the forward projection $\mathrm{FP}_{\mathbf{CS}}(\mathcal{P})$ is completely contained in $\mathcal{G}$. Hence, the point A and any configuration outside $\mathcal{G}$ are **CS**-$\mathcal{P}$-distinguishable, so that A belongs to $\chi_{\mathbf{CS},\mathcal{P}}(\mathcal{G})$. The same kind of verification can be extended to the other vertices B through H, the straight and circular edges connecting these points, and the interior of the outlined area. The resulting preimage $\mathcal{P}$ is the union of $\chi_{\mathbf{CS},\mathbf{TC}}(\mathcal{G})$ and the truncated cone of angle $2\eta_v$ on top of it (Figure 2.3). It is larger than that shown in Figure 2.2(b) - the preimage without state. A similar construction was given by Erdmann [Erd 84], but it was not defined precisely. The region outlined in a dashed line depicts the set of sensed configurations for which the termination condition $\mathcal{K}_{\mathbf{CS},\mathcal{P}}(\mathbf{q}^*) \subseteq \mathcal{G}$ evaluates to `true`.$\diamond$

2.4.3 Interpretation and Preimage with Initial and Final States

Interpretations with Initial and Final States

Any condition **TC** divides a forward projection $\mathrm{FP}(\mathcal{I})$ into three regions, which we denote by F_1, F_2 and F_3:

- F_1 consists of all the states $\mathbf{s} \in \mathrm{FP}(\mathcal{I})$ such that, for every possible sensory data $\mathbf{m}^* \in \mathcal{K}^*(\mathbf{s})$, **TC** evaluates to `false`,

- F_2 consists of all the states $\mathbf{s} \in \mathrm{FP}(\mathcal{I})$ such that, for every possible sensory data $\mathbf{m}^* \in \mathcal{K}^*(\mathbf{s})$, **TC** evaluates to `true` (recall condition (i) in the definition of *Achieve* in equation 1.2), and

- $F_3 = \mathrm{FP}(\mathcal{I})\backslash(F_1 \cup F_2)$, i.e. any state in F_3 may non-deterministically produce sensory data $\mathbf{m}^* \in \mathcal{K}^*(\mathbf{s})$, for which **TC** evaluates to either `true` or `false` (recall condition (ii) in the definition of *Achieve* in equation 1.2).

At every instant during a motion commanded according to **CS** and issued from within $\mathcal{I}$, if the current state is in F_1, the motion keeps going; if it is in F_2, the motion stops; if it is in F_3, the motion may either continue or stop. In general, there exist subsets of $\mathrm{FP}(\mathcal{I})$ that are inaccessible from $\mathcal{I}$ because reaching them would require to previously traverse F_2, where the motion would have been terminated. This is precisely why making the termination condition know itself increases its recognition power.

Given: a termination condition **TC**, a motion commanded according to **CS**, and starting from within $\mathcal{I}$, a state $\mathbf{s}_a$ can be reached iff there exist a configuration $\mathbf{s}_s \in \mathcal{I}$ and a trajectory in $\Phi(\mathbf{s}_s)$ which connects $\mathbf{s}_s$ to $\mathbf{s}_a$ without traversing F_2, except possibly at $\mathbf{s}_a$ itself. Let us denote the set of all such states $\mathbf{s}_a$ by $\mathcal{F}_{\mathbf{CS},\mathbf{TC}}(\mathcal{I}) \subseteq \mathrm{FP}(\mathcal{I})$. Given $\mathcal{I}$, $\mathcal{F}_{\mathbf{CS},\mathbf{TC}}(\mathcal{I})$ is the *Final State* of the system [SL 91, LLS 91].

So, given a sensing query $\mathbf{m}^*$, all interpretations that are mutually consistent, consistent with the initial state, and the final state denoted $\mathcal{K}_{\mathbf{CS},\mathcal{I},\mathbf{TC}}(\mathbf{m}^*)$, are defined as

$$\mathcal{K}_{\mathbf{CS},\mathcal{I},\mathbf{TC}}(\mathbf{m}^*) = \mathcal{K}_{\mathbf{CS},\mathcal{I}}(\mathbf{m}^*) \bigcap \mathcal{F}_{\mathbf{CS},\mathbf{TC}}(\mathcal{I}).$$

Preimage with Initial and Final State

Two configurations $\mathbf{s}_1$ and $\mathbf{s}_2$ in CQ_{valid} are said to be **CS-$\mathcal{P}$-TC-distinguishable** iff:

$$\{\mathbf{m}^* \;/\; \mathbf{s}_1, \mathbf{s}_2 \in \mathcal{K}_{\mathbf{CS},\mathcal{P},\mathbf{TC}}(\mathbf{m}^*)\} = \emptyset.$$

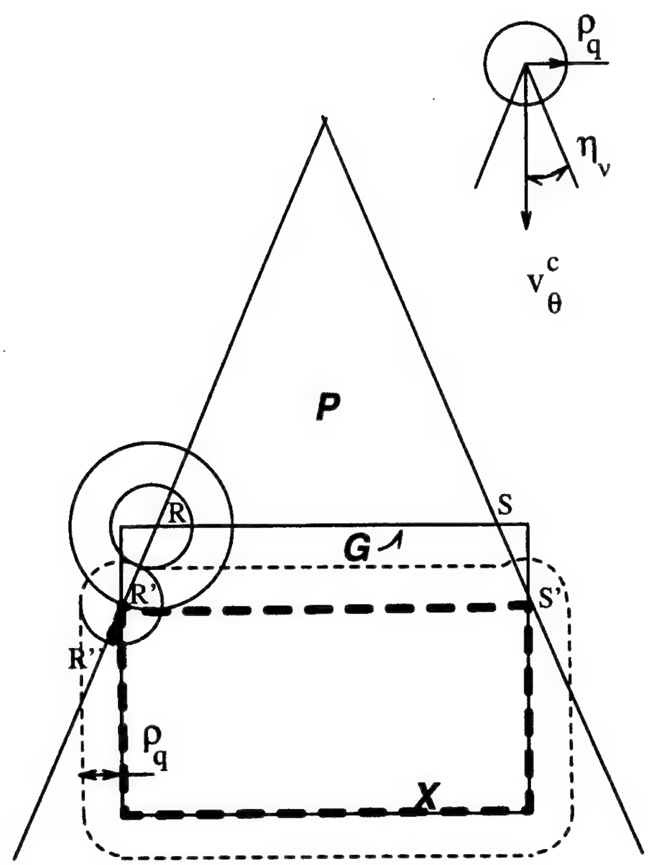


Figure 2.4: Using Initial and Final States

The **CS-$\mathcal{P}$-TC-kernel** of a goal $\mathcal{G}$ is defined as:

$\chi_{\mathbf{CS},\mathcal{P},\mathbf{TC}}(\mathcal{G}) = \{\mathrm{s} \in \mathcal{G} \ / \ \forall \mathrm{s}' \in \mathcal{Q}_{valid} \backslash \mathcal{G} : \mathrm{s} \text{ and } \mathrm{s}' \text{ are } \mathbf{CS}-\mathcal{P}-\mathbf{TC}-\text{distinguishable}\}$

$\mathcal{P}$ is a preimage of $\mathcal{G}$ for the motion command $(\mathbf{CS}, \mathbf{TC})$, with:

$$\mathbf{TC} \quad \equiv \quad \mathcal{K}_{\mathbf{CS},\mathcal{P},\mathbf{TC}}(\mathbf{m}^*) \subseteq \mathcal{G},$$

iff it is equal to the backprojection of the **CS**-$\mathcal{P}$-**TC**-kernel of $\mathcal{G}$ for the control statement **CS**, *i.e.*, $\mathcal{P} = \mathcal{B}_{\mathbf{CS}}(\chi_{\mathbf{CS},\mathcal{P},\mathbf{TC}}(\mathcal{G}))$. The above condition **TC** is called a termination condition with initial and **final** states [SL 91, LLS 91].

Example 2.4 We show below that using a termination condition with initial and final states makes it possible to construct a preimage larger than that of Figure 2.3.

The region with thick dashed shown in Figure 2.4 is a generalized polygon constructed as follows. Let R (resp. S) be the points in the top horizontal edge of $\mathcal{G}$ such that the intersection of $\mathcal{G}$ and a line passing through R (resp. S) and parallel to the left (resp. right) side of the control uncertainty cone is a segment RR' (resp. SS') of length $2\rho_q$. The line $R'S'$ that forms the upper portion of the thick dashed contour consists of two circular arcs of radius $2\rho_q$, with respective centers R and S, and a straight segment at distance $2\rho_q$ from the top edge of the goal $\mathcal{G}$. The rest of the thick dashed contour is the lower part of $\mathcal{G}$'s boundary. The region thus outlined is the kernel $\chi_{\mathbf{CS},\mathcal{P},\mathbf{TC}}(\mathcal{G})$ with $\mathcal{P} = \mathcal{B}_{\mathbf{CS}}(\chi_{\mathbf{CS},\mathcal{P},\mathbf{TC}}(\mathcal{G}))$ and $\mathbf{TC} \equiv [\mathcal{K}_{\mathbf{CS},\mathcal{P},\mathbf{TC}}(\mathbf{m}^*) \subseteq \mathcal{G}]$. The region outlined in a thin dashed line depicts the set of sensed configurations for which the termination condition evaluates to `true`.

One can verify that the region $\mathcal{P}$ is a preimage of $\mathcal{G}$. Indeed, all the motions starting from within $\mathcal{P}$ are guaranteed to reach the goal (if they are not terminated before) and none of them can be terminated before the goal has been attained. In addition, none of these motions can leave the goal without being terminated.

In order to see how the termination condition uses the knowledge of itself, assume that at some instant during the motion the actual configuration is the point designated by R' in figure 2.4. All the possible sensed configurations at this instant lie in the disc of radius ρ_q centered at R'. If neither the forward projection nor the termination condition are taken into account, the set of all the interpretations of all these sensed configurations is a disc of radius $2\rho_q$ centered at R'. The intersection of this disc with the forward projection $\mathrm{FP}_{\mathbf{CS}}(\mathcal{P})$ contains a sector that is not contained in $\mathcal{G}$. This sector (the grey sector in Figure 2.4) can only be attained from $\mathcal{P}$ by crossing the segment marked $R'R''$. Since for any configuration in this segment the termination condition evaluates to `true`, the sector cannot be attained. The preimage built in Figure 2.4 is substantially larger than the one in Figure 2.3.$\diamond$

Chapter 3

Control Uncertainty

We now consider the problem of computing the *boundary* of a forward projection, and a backprojection, and the boundary of perturbations of the stable and unstable manifolds of singularities on smooth sets of an arbitrary, but autonomous control scheme. The control equations include state-dependent differential equations governing general rigid body motion either in free space or in contact space. Section 1.2.1 considered one such example of a second-order equations of motion as a vector field on a smooth manifold TM of constant dimension. Here, M itself is a smooth manifold containing a strata $\mathcal{S}_i$ of $\mathcal{Q}_{valid}$. For simplicity of notation, however, we use M to represent the tangent bundle TM. Therefore, it is understood that for second-order equations, M is twice the dimension of the strata.

The first section reviews results from Differential Inclusion. This review is presented on a standard Euclidean space. Other sections consider manifold structure.

3.1 Differential Inclusion Problem

Consider the multivalued map F^0 in Proposition 2.2. At every point $\mathrm{s} \in M$, its image is a subset of the tangent space $T_{\mathrm{s}}M$. Graphically, if this region is thought of as a neighborhood of a nominal vector field X, it contains the tip of the vector $X(\mathrm{s})$ and a region around the tip $X(\mathrm{s})$.

It is sufficient to assume that the problem of differential inclusion is given on

Euclidean spaces. For $\mathbf{s} \in U \subset \mathbf{R}^n$, U a compact subset of $\mathbf{R}^n$, consider the *Cauchy* problem of finding all *absolutely continuous* functions $\phi: \mathbf{R} \times U \rightarrow \mathbf{R}^n$

$$\begin{aligned} \phi(0, \mathbf{s}) &= \mathbf{s}, \\ D_1\phi\left(\frac{\partial}{\partial t}\Big|_{(t,\mathbf{s})}\right) &\in F^0(\phi(t, \mathbf{s})), \quad \textit{almost everywhere.} \end{aligned} \tag{3.1}$$

This is a problem in *Differential Inclusion* and a direct generalization of an ordinary differential equation such as equation 2.23. Questions in ordinary differential equations such as existence of solutions, continuity of solutions, continuous dependence on initial conditions have a natural counterpart in differential inclusion. These problems of differential inclusion were studied extensively since 1930's. For a historical account and the results, refer to a survey article by Blagodat·skikh and Filippov [BlFi 86] and a book by Aubin and Cellina [AubCel 84].

3.1.1 Calculus of Multivalued Maps

Let upper case letters $F, G, \ldots$ denote subsets of $\mathbf{R}^n$. Define the distance between two subsets F and G to be $d(F, G) = \inf_{\mathbf{s} \in F, \mathbf{s}' \in G} d(\mathbf{s}, \mathbf{s}')$, where $d(\mathbf{s}, \mathbf{s}')$ is the standard Euclidean distance between points. We now describe continuity and Lipschitz continuity of multivalued maps $F: U \rightarrow \mathbf{R}^n$. Continuity of a multivalued map follows two different but equivalent notions of continuity of a single-valued map. They are:

1. For any neighborhood $\mathbf{N}(F(\mathbf{s}))$ of $F(\mathbf{s})$, there exists a neighborhood $\mathbf{N}(\mathbf{s})$ of $\mathbf{s}$ such that $F(\mathbf{N}(\mathbf{s})) \subset \mathbf{N}(F(\mathbf{s}))$.

2. Any generalized sequence $\mathbf{s}_k$ converging to $\mathbf{s}$, and for any $\mathbf{v}_\mathbf{s} \in F(\mathbf{s})$, there exists a sequence $\mathbf{v}_{\mathbf{s}_k} \in F(\mathbf{s}_k)$ that converges to $\mathbf{v}_\mathbf{s}$.

Although these properties are equivalent for a single-valued map, they correspond to *upper semicontinuous* and *lower semicontinuous* multivalued maps, respectively. Consider the following two examples of set valued maps:

1. The mapping F from $\mathbf{R}$ to its subset defined by $F(0) = [-1, 1]$ and $F(\mathbf{s}) = \{0\}, \mathbf{s} \neq 0$ is an upper semicontinuous map, but not lower semicontinuous.

2. The mapping F from $\mathbf{R}$ to its subset defined by $F(0) = \{0\}$ and $F(s) = [-1,1], s \neq 0$ is a lower semicontinuous map, but not upper semicontinuous.

A multivalued map is *continuous* if it is both upper and lower semicontinuous. Corresponding to Lipschitz single-valued maps that are continuous but not continuously differentiable, there exists *Lipschitz multivalued maps.* First define an ϵ-neighborhood of a subset $K \subset \mathbf{R}^n$ by $B(K,\epsilon) = \{\mathbf{v_s} \in \mathbf{R}^n | d(\mathbf{v_s}, K) < \epsilon\}$. A multivalued map is *Locally Lipschitz* at $\mathbf{s}_0$ if there exists a neighborhood $\mathbf{N}(\mathbf{s}_0)$ of $\mathbf{s}_0$ and a constant $L > 0$ called the Lipschitz constant such that

$$\forall \mathbf{s}', \mathbf{s} \in \mathbf{N}(\mathbf{s}_0), \qquad F(\mathbf{s}) \subset B(F(\mathbf{s}'), L\, d(\mathbf{s}, \mathbf{s}')). \tag{3.2}$$

The multivalued map is *Lipschitzian* if

$$\forall \mathbf{s}', \mathbf{s} \in U, \qquad F(\mathbf{s}_0) \subset B(F(\mathbf{s}'), L\, d(\mathbf{s}_0, \mathbf{s}')). \tag{3.3}$$

A C^0 neighborhood F^0 of a vector field is a multivalued map that assigns to each point in the manifold a set of tangent vectors. It is apparent that if the boundary of neighborhood F^0 is continuous, or Lipschitz, the multivalued map is respectively continuous, or Lipschitz.

3.1.2 Selection Scheme for a Multivalued Map

The selection problem for a multivalued map $F: U \to \mathbf{R}^n$ is finding a single-valued map $f: U \to \mathbf{R}^n$ such that $f(\mathbf{s}) \in F(\mathbf{s})$ for all points in U (when the base set U is a finite set, the problem is same as the *Axiom of Choice*). In general, there may not exist a continuous selection when the multivalued map F is continuous. If a prescribed rule for selection is specified, such as minimal selection or Chebishev selection, the selection can fail to be Lipschitz even when the map F is Lipschitz. See examples in Aubin and Cellina's book [AubCel 84].

Consider the selection problem for a multivalued map $F^0: U \to \mathbf{R}^n$. A parametrization of the multivalued map is a selection scheme such that a single-valued map $Y: \mathcal{D}^n \times U \to \mathbf{R}^n$ is equivalent to F^0 where $F^0(\mathbf{s}) = \{Y(\mathbf{p}, \mathbf{s}) | \mathbf{p} \in \mathcal{D}^n\}$ for $\mathcal{D}^n$, an n-dimensional closed disk. In the following, we consider the parametrization problem for the boundary δF of the multivalued map F.

First, define the *support function* of a non-empty subset $F \subset \mathbf{R}^n$ for a vector $\psi \in \mathbf{R}^n$ as a scalar function $c(F,\psi) = sup_{\mathbf{v}\in F}(\psi \cdot \mathbf{v})$. As an example consider the support function of a ball $B_r(\mathbf{a})$ of radius r with center at $\mathbf{a}$ as $c(B_r(\mathbf{a}),\psi) = \mathbf{a}\cdot\psi + r|\psi|$. Several properties of a support function necessary to define its differential are available in the article by Blagodat·skikh and Filippov [BlFi 86]. In particular, a support function is convex with respect to ψ if $c(F, \alpha\psi_1 + \beta\psi_2) \leq \alpha\, c(F,\psi_1) + \beta\, c(F,\psi_2)$ for $\alpha, \beta \geq 0$ and $\psi_1, \psi_2 \in \mathbf{R}^n$. Define the subdifferential of a support function $c(F,\psi)$ with respect to ψ as $\partial c(F,\psi_0) = \{\mathbf{v} \in \mathbf{R}^n | \mathbf{v}\cdot(\psi - \psi_0) \leq c(F,\psi) - c(F,\psi_0), \psi \in \mathbf{R}^n\}$ The subdifferential of the support function $c(B_r(\mathbf{a}),\psi)$ is

$$\partial c(B_r(\mathbf{a}),\psi) = \begin{cases} \mathbf{a} + r\frac{\psi}{|\psi|}, & \text{if } \psi \neq 0; \\ \mathbf{a} + B_1(0), & \text{if } \psi = 0. \end{cases}$$

If the support function is finite, it is differentiable with respect to ψ at ψ_0 if and only if the subdifferential $\partial c(F,\psi_0)$ is single-valued. A multivalued map F is *strictly convex in the direction* ψ_0 iff the support function $c(F,\psi_0)$ is differentiable. It is *strictly convex* iff the support function $c(F,\psi)$ for $\forall\psi \neq 0$ is differentiable. As a consequence of strict convexity of a multivalued map F, it is possible to construct a parametrization of the boundary δF namely

$$\partial c(F,\psi)/\partial\psi \colon \mathbf{S}^{n-1} \to \mathbf{R}^n$$

which is continuous. An example is the subdifferential $\partial c(B_r(\mathbf{a}),\psi)/\partial\psi$ of a ball $B_r(\mathbf{a})$, given earlier. We later impose that this parametrization of the boundary is a smooth or piecewise C^1 differentiable. With the assumption that F^0 as a multivalued map is strictly convex with finite support function, we denote the corresponding parametrized field as

$$Y \colon S^{n-1} \times U \to \mathbf{R}^n \colon (\mathbf{p},\mathbf{s}) \mapsto Y(\mathbf{p},\mathbf{s}) \in \delta F(\mathbf{s}). \tag{3.4}$$

3.1.3 Flows of Bounded Perturbations of Vector Fields

Consider the space of absolutely continuous functions $AC(I,\mathbf{R}^n)$ with $I = [0,T]$, $T > 0$ and a norm $||\phi||_{AC} = \phi(0) + \int_0^T |\phi(\tau)|\, d\tau$ on $AC(I,\mathbf{R}^n)$. Consider $\Phi(\mathbf{s}) \subset AC(I,\mathbf{R}^n)$

the set of all solutions to the Cauchy problem 3.1. This denotes all absolutely continuous functions of time defined at $\mathbf{s}$ which are solutions of the differential inclusion problem. The following is a basic result established by Cellina and Ornelas [CelOrn] on path connectedness of the set of solutions of the Cauchy problem.

Theorem 3.1 *Let the multivalued map F be non-empty, closed, Lipschitzian, and autonomous. The selection map from U to $AC(I, \mathbf{R}^n)$ that assigns to any point $\mathbf{s}$ a solution $\phi_{\mathbf{s}} \in \Phi(\mathbf{s})$ is continuous.*

As a corollary, they also establish that the solution set $\Phi(\mathbf{s})$ and the attainable set $\mathcal{A}(t, \mathbf{s})$ can be continuously parametrized, and in particular they are analytical sets.

Corollary 3.1 *There exists closed subset $\mathcal{U}$ of a separable Banach space of continuous maps from the set U to the space of absolutely continuous functions $AC(I, \mathbf{R}^n)$ and continuous function $g: U \times \mathcal{U} \rightarrow AC$ such that $g(\mathbf{s}, \mathcal{U}) = \Phi(\mathbf{s})$.*

Corollary 3.2 *There exists closed subset $\mathcal{U}$ of a separable Banach space and continuous function $h: U \times \mathcal{U} \rightarrow \mathbf{R}^n$ such that $h(\mathbf{s}, \mathcal{U}) = \mathcal{A}(t, \mathbf{s})$.*

Consider another result from Staicu and Wu [StWu 91] that proves that any solution in the solution set can be deformed to another.

Theorem 3.2 *Let F be a non-empty, closed, Lipschitzian, and autonomous multivalued map on the set U. Let $\mathbf{s} \rightarrow \phi^0_{\mathbf{s}}$ and $\mathbf{s} \rightarrow \phi^1_{\mathbf{s}}$ be two selection from $\mathbf{s} \rightarrow \Phi(\mathbf{s})$ continuous from U to $AC(I, \mathbf{R}^n)$. Then, there exists a map H from $[0,1] \times U \rightarrow AC(I, \mathbf{R}^n)$ with the following properties:*

1. *H is continuous.*
2. *$H(0, \mathbf{s}) = \phi^0_{\mathbf{s}}$, and $H(1, \mathbf{s}) = \phi^1_{\mathbf{s}}$.*
3. *For $\lambda \in [0,1]$, $H(\lambda, \mathbf{s})$ is in $\Phi(\mathbf{s})$.*

This result together with the continuity of the selection implies that the solution set $\Phi(\mathbf{s})$ is path connected in the space of absolutely continuous functions $AC(I, \mathbf{R}^n)$ with the topology induced by the norm.

A theorem from Aubin and Cellina [AubCel 84] asserts that the solution set of a Cauchy problem for non-convex sets in the right-hand side of equation 3.1 is dense in the solution set of the convex hull of the non-convex set.

Theorem 3.3 (Relaxation Theorem) *If the multivalued map F^0 is Lipschitzian, the set of solutions to the Cauchy problem in equation 3.1 is dense in the set of trajectories of the Cauchy problem*

$$\begin{aligned} \phi(0,\mathbf{s}) &= \mathbf{s} \\ D_1\phi\left(\frac{\partial}{\partial t}\Big|_{(t,\mathbf{s})}\right) &\in \mathrm{co}(F^0(\phi(t,\mathbf{s}))), \quad \text{almost everywhere} \end{aligned} \tag{3.5}$$

where co(F) *denotes convex hull of the set F.*

It is sufficient to consider $co(F^0)$ if we assume that the set valued map F^0 is Lipschitzian. Strict convexity of F^0 is also necessary for parametrization of the boundary of the multivalued map as in equation 3.4. So, from now onwards we only consider those multivalued maps F^0 that are strictly convex and Lipschitzian.

In summary, if we assume that the multivalued map F^0 is non-empty, and strictly convex, then the boundary of the multivalued map is parametrizable. If it is also Lipschitzian, and autonomous, and the domain is a compact subset, then the solution set of any point is path connected in the space of absolutely continuous flows. It also follows that the solution set of any path connected set is also path connected.

3.2 Boundary of Control Uncertainty

A possible description of all flows $\Phi(\mathbf{s})$ are the set of states expressed either as a function of time called the *attainable set* or the time-independent set called *forward projection* defined in Section 2.2.3. These are multivalued maps from the phase space to itself. As a consequence of Corollary 3.1 and 3.2, the *attainable set* and the *forward projection* admit representation by their boundary. Denote the topological boundary of the attainable set at time t as $\delta\mathcal{A}(t,\mathbf{s})$, and that of the forward projection as $\delta\mathrm{FP}(\mathbf{s})$. In the following sections, we present necessary conditions for the characterization of the boundary of the attainable set and the forward projection.

If the differential inclusion problem is specified on compact manifolds, then each phase flow $\phi_{\mathbf{s}}$ is defined for all $t \in \mathbf{R}$. However, since we consider compact subsets with boundary, the solutions $\phi_{\mathbf{s}} \in \Phi(\mathbf{s})$ may be defined for unequal intervals of time. Denote $\mathcal{J}_{\phi_{\mathbf{s}}}$ as the interval of time for which a solution $\phi_{\mathbf{s}}$ is defined.

3.2.1 Boundary of Attainable Set

The attainable set defined in Section 2.2.3 is a multivalued map $\mathcal{A}(t, \mathbf{s})\colon U \rightarrow \mathbf{R}^n$. The following theorem from Blagodat·skikh and Filippov [BlFi 86] characterizes $\delta\mathcal{A}$, the boundary of the attainable set.

Theorem 3.4 *Assume that the support function $c(F^0, \psi)$ of the set $F^0(\mathbf{s})$ is continuously differentiable with respect to ψ and the vector $\partial c(F^0, \psi)/\partial\psi$ and $\partial c(F^0, \psi)/\partial\mathbf{s}$ are Lipschitz functions with respect to ψ and $\mathbf{s}$. Consider the $2n$-system of differential equations*

$$\dot{\mathbf{s}} = \frac{\partial c(F^0, \psi)}{\partial \psi}; \ \dot{\psi} = -\frac{\partial c(F^0, \psi)}{\partial \mathbf{s}}. \tag{3.6}$$

Then, for $0 \leq t_1 \leq t^*$, the boundary $\delta\mathcal{A}(t_1, \mathbf{s}_0)$ of the set of attainability coincides with the set $\Gamma(t_1, \mathbf{s}_0)$ of all points $\phi(t_1, \mathbf{s}_0)$, where the pair $(\phi(t, \mathbf{s}), \psi(t))$ is a solution of the system 3.6 with the initial condition

$$\phi(t_0, \mathbf{s}_0) = \mathbf{s}_0, \ \psi(t_0) \in S, \tag{3.7}$$

and $S = \{\psi \mid |\psi| = 1\}$ is the unit sphere in $\mathbf{R}^n$. The inclusion $\delta\mathcal{A}(t, \mathbf{s}_0) \subset \Gamma(t, \mathbf{s}_0)$ holds for $t > t^*$. The time t^* is the infimum of the times t' such that $\phi_1(t', \mathbf{s}_0) = \phi_2(t', \mathbf{s}_0)$ for two solutions (ϕ_1, ψ_1) and (ϕ_2, ψ_2) of system 3.6 with non-coinciding initial conditions of the form 3.7.

The attainable set may not be closed, see [CelOrn].

3.2.2 Boundary of Forward Projection

The Forward Projection on smooth sets defined in Section 2.2.3 is a multivalued map $\mathrm{FP}\colon U \rightarrow M$ where U is a compact subset of a manifold M. It is the union over

time of the attainable set. Firstly, flow functions $\phi_s \in \Phi(s)$ are absolutely continuous functions of time. As a consequence of path connectedness of the attainable set $\mathcal{A}(t, s)$ [StWu 91], the Forward Projection, FP(s), is path-connected. Therefore, with additional conditions on the multivalued map F^0, we can hope to find a reasonably well-behaved boundary of the forward projection.

In computing the boundary of the forward projection, we compute admissible trajectories on the boundary. For the one-dimensional case, the boundary of the attainable set is defined by two trajectories - intuitively the fastest and the slowest moving particles that start at the point. An explicit integration of two ordinary differential equations, therefore, represents the boundary. The forward projection is bounded by the initial point and the fastest moving component of the attainable set. In higher dimensions, the boundary is computed as a solution of a *Hamilton-Jacobi* problem - equivalent to solving a scalar partial differential equation. It describes the boundary of the forward projection outside regions we call *Singular Invariant Subsets*, which are subsets where differential inclusion specification F^0 contains singularities, *i.e.,* $0 \in F^0$. The first section computes the boundary of the *Singular Invariant Subsets* and gives the condition when such a boundary can be uniformly labeled as a singularity of type *source*, *sink*, or *saddle.* Outside such regions, the boundary of the forward projection is described by the solution of the Hamilton-Jacobi problem. If a singular invariant subset of the saddle type is present, then the boundary of the forward projection must also contain a subset of the boundary of perturbation of *unstable manifolds* of the saddle. Such additional components are present in the description of the forward projection of points that lie in the perturbation of the *stable manifolds* of the saddle. We present characterization of the boundary of perturbations of the stable and unstable manifolds of a saddle in another section.

The Hamilton-Jacobi Theorem allows one to solve scalar partial differential equations by solving ordinary differential equations - known as the *method of characteristic* in partial differential equation literature. Historically, a scalar non-linear partial differential equation has also been called a *Monge's Cone* [Tr 57, John 82]. This method of characteristic is also used by Butkovskii [But 82] in computing admissible trajectories in problems of controllability and finite control.

3.2.3 Singular Invariant Subset

If the orbit of all possible absolutely continuous functions that are solutions to the Cauchy differential inclusion problem remain in the subset, then the subset is a *strongly invariant* subset. If there exists at least a solution such that its orbit remains in the subset, then it is a *weakly invariant* subset. Consider a point $\mathbf{s}$ in M such that $0 \in F^0(\mathbf{s})$. A possible flow is the trivial one - a constant map, $\phi(t, \mathbf{s}) = \mathbf{s}$, defined for all times. Such points are, therefore, at least weakly invariant.

Definition 3.1 *The Singular invariant subset*

$$\mathcal{Z}_i = \{\mathbf{s} \in M | 0 \in F^0(\mathbf{s})\}$$

consists of all points on the manifold such that there exists a vector field in the neighborhood F^0 with a singularity at the point. The union $\cup_i \mathcal{Z}_i$ of Singular invariant subsets of M is denoted $\mathcal{Z}$.

If a multivalued map F^0 is given as a neighborhood of some nominal vector field X, and $X(\mathbf{s}) = 0$ for some $\mathbf{s}$, the region $F^0(\mathbf{s})$ definitely contains zero. For other $\mathbf{s}'$ nearby $\mathbf{s}$, when $X(\mathbf{s}') \neq 0$, the region $F^0(\mathbf{s}')$ may still contain zero since $(-X(\mathbf{s}'))$ is in $F^0(\mathbf{s}')$.

Singular Invariant subsets are closed subsets of manifold M because the multivalued map F^0 is closed. Each of the connected components of a singular invariant subset is labeled $\mathcal{Z}_i$. Consider the boundary δF^0 of C^0 neighborhood F^0. With conditions that give the boundary δF^0 a parametrization in equation 3.4, the map Y is a well-defined map from the $(n-1) + n (= 2n-1)$ dimensional manifold $S^{n-1} \times M$ to TM, the $2n$-dimensional tangent bundle of M.

Proposition 3.1 *The singular invariant set is implicitly defined by the neighborhood F^0 if the parametrized boundary field Y is a transversal section of the tangent bundle $\tau = (TM, M, \tau)$ for each parameter $\mathbf{p} \in S^{n-1}$.*

Proof: The proof follows from a direct application of Implicit Function Theorem. In the following, all constructions are based on local tangent bundle charts. Consider

the local tangent bundle chart (U_k, ϕ_k) of the tangent bundle $\tau = \{TM, M, \tau\}$ and a chart (V_j, η_j) of S^{n-1} around a point $(\mathbf{p}, \mathbf{s}) \in V_j \times U_k$, where $Y(\mathbf{p}, \mathbf{s}) = 0$. The map Y induces a corresponding map Y' on the local tangent bundle chart

$$Y': \eta_j V_j \times \phi_k U_k \to \phi_k U_k \times \mathbf{R}^n: (\eta_j(\mathbf{p}), \phi_k(\mathbf{s})) \mapsto (\phi_k(\mathbf{s}), Y'(\eta_j(\mathbf{p}), \phi_k(\mathbf{s})))$$

The derivative of this map is

$$DY' = \begin{bmatrix} 0 & id_{n\times n} \\ D_1 Y' & D_2 Y' \end{bmatrix} \tag{3.8}$$

Let σ be a projection map defined as $\sigma: \phi_k U_k \times \mathbf{R}^n \to \mathbf{R}^n: (\phi_k(\mathbf{s}), Y'(\eta_j(\mathbf{p}), \phi_k(\mathbf{s}))) \mapsto Y'(\eta_j(\mathbf{p}), \phi_k(\mathbf{s}))$. The derivative of the composite map $\sigma \circ Y'$ is

$$D(\sigma \circ Y') = [\, 0 \quad id_{n\times n} \,] \begin{bmatrix} 0 & id_{n\times n} \\ D_1 Y' & D_2 Y' \end{bmatrix} = [\, D_1 Y' \quad D_2 Y' \,]$$

Since, Y is a transversal section for each $\mathbf{p}$, $D_2 Y'$ is full rank. By the Implict Function Theorem, the condition $\sigma \circ Y'(\eta_j(\mathbf{p}), \phi_k(\mathbf{s})) = 0$ defines a continuous function $g'_{jk}: \eta_j V_j \to \phi_k U_k$ such that $Y'(\eta_j(\mathbf{p}), g'_{jk}(\eta_j(\mathbf{p}))) = 0$. Define the composition map $g_{jk} = \eta_j \circ g'_{jk} \circ \phi_k^{-1}: V_j \to U_k$. Piecing all these g_{jk}'s over different charts, a global function $g_i: S^{n-1} \to M$ is constructed such that $Y(\mathbf{p}, g_i(\mathbf{p})) = 0$. $\diamond$

The function g_i is the required map implicitly defining the boundary of the singular invariant set. In principle, the field Y may have several such regions and the boundary of each of such region $\delta \mathcal{Z}_i$ is defined by a unique g_i.

Proposition 3.2 *Each singular invariant subset $\mathcal{Z}_i$ is disjoint when the parametrized vector field Y satisfies conditions given in the previous proposition.*

Proof: Consider two such regions $\mathcal{Z}_{i_1}$ and $\mathcal{Z}_{i_2}$. They are regular subsets of M, and hence if they have a point in common, so must their boundary. The boundary of such regions are described by maps $g_{i_1}, g_{i_2}: S^{n-1} \to M$. They can not have a point $\mathbf{s}_0 \in M$ in common since functions g_{i_1} and g_{i_2} are unique maps from S^{n-1} to M from the previous proposition. $\diamond$

In addition, if the parametrized vector field Y is a hyperbolic section for each parameter then the following proposition shows that each singular invariant subset can be characterized.

Proposition 3.3 *Each singular invariant subset $\mathcal{Z}_i$ can be characterized as being of type sink, source, or saddle, if the parametrized boundary field Y is a hyperbolic section and C^1 in variables $\mathbf{p}$ and $\mathbf{s}$.*

Proof: Consider implicitly defined function $g_i: S^{n-1} \rightarrow M$ in an earlier proposition such that $Y(\mathbf{p}, g_i(\mathbf{p})) = 0$. For any point $\mathbf{p}_0 \in S^{n-1}$, the linear map $D_2Y(\mathbf{p}_0, g_i(\mathbf{p}_0))$ is either of the *sink*, *source*, or *saddle* type. We claim that this is invariant to the choice of point $\mathbf{p}_0$. Since Y is C^1 section, the derivative D_2Y changes continuously on the sphere S^{n-1}. Without proving, we state that the eigenvalues of a linear operator depend continuously on the operator. The eigenvalues of D_2Y change continuously on the sphere. If $D_2Y(\mathbf{p}_0, g_i(\mathbf{p}_0))$ is hyperbolic, all eigenvalues have non-zero real parts. Since, eigenvalues change continuously and have non-zero real parts for all points on the sphere S^{n-1}, the sign of the real parts do not change at any point. Hence all points have the same characteristic spectrum. $\diamond$

If the vector field is not hyperbolic, the singular invariant subsets may not be characterized as above. However, consider the following observation [PalMel 82]: the set of all hyperbolic vector fields among the set of all C^1 vector fields forms an open and dense set. Hence, any non-hyperbolic field can be perturbed by an arbitrary small amount to obtain a hyperbolic field. In light of this, our restriction to only hyperbolic fields covers *almost all* fields. Recall that this restriction is placed on the parametrized boundary field Y - any interior field needs to be merely integrable.

3.2.4 The Cone Boundary Surface - Hamilton-Jacobi Form

A *cone* $C(\mathbf{s})$ with vertex at $(\mathbf{s}, 0)$ is defined to be a subset of $T_\mathbf{s}M$ such that if a point $(\mathbf{s}, \mathbf{v_s})$ is in this subset, then so are points $(\mathbf{s}, \lambda\mathbf{v_s}), \lambda > 0$. Consider the C^0 neighborhood map F^0 with conditions that give it a parametrization as a vector field Y in equation 3.4. Consider the cone $C(\mathbf{s})$ given by all half-straight-lines passing through $(\mathbf{s}, 0)$ and points in $F^0(\mathbf{s})$. Since set $F^0(\mathbf{s})$ is convex, the cone is also a convex cone. The elements in $\delta C(\mathbf{s}) \in F^0(\mathbf{s})$ are defined to be the *Cone Boundary Field* and a *Dual Cone* is defined by

$$C^*(\mathbf{s}) = \{\mathbf{w}^*_\mathbf{s} \in T^*_\mathbf{s}M | \mathbf{w}^*_\mathbf{s}(\mathbf{v_s}) \leq 0, \forall \mathbf{v_s} \in F^0(\mathbf{s})\} \tag{3.9}$$

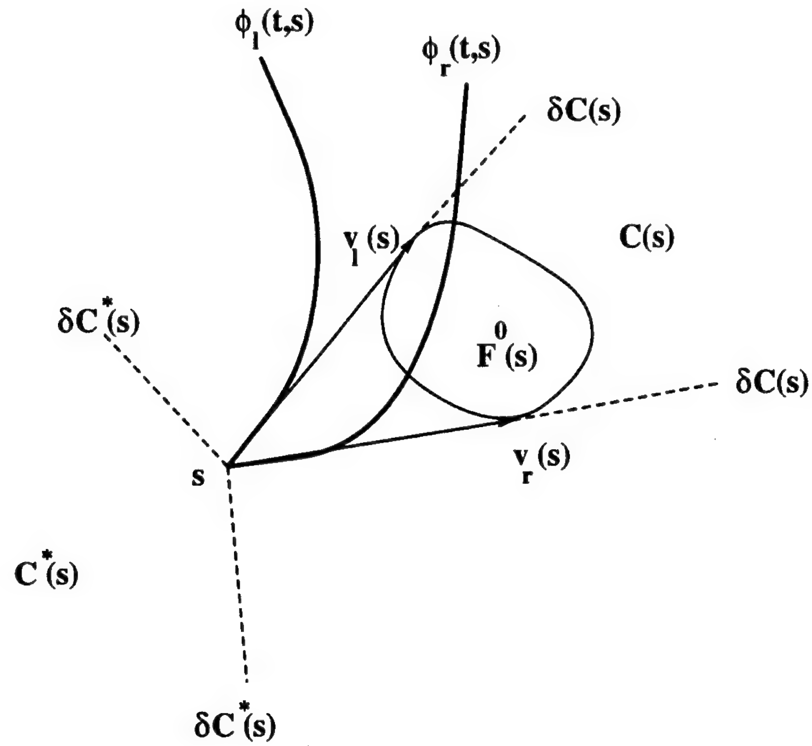


Figure 3.1: A Canonical Two-dimensional Example of a Cone Boundary Field

All points $\mathbf{s} \in M \setminus \mathcal{Z}$ have a non-empty dual cone. For points in $\mathcal{Z}$, the cone $C(\mathbf{s})$ spans the complete space $T_{\mathbf{s}}M$, since $0 \in F^0$. It is easy to see that the dual cone is empty at such points. The topological boundary of the cones is denoted $\delta C(\mathbf{s})$ and $\delta C^*(\mathbf{s})$ (see Figure 3.1).

After a brief overview of Hamiltonian systems, we give constructions necessary to compute the boundary of the forward projection. For details on Hamiltonian systems, see Abraham and Marsden [AbMa 78]. Consider manifold T^*M which has a *canonical symplectic* structure given by a *non-degenerate*, *closed* two form $\omega \in \Lambda^2(T^*M)$. Darboux's Theorem [AbMa 78] guarantees the existence of a *canonical form* $\omega = \sum ds^i \wedge dp_j$ for any non-degenerate, closed two form ω. The corresponding coordinates (s^i, p_i) on T^*M are called *canonical symplectic coordinates.*

Definition 3.2 *A* Hamiltonian Vector Field X_H *on* T^*M *with symplectic structure*

ω *for a* Hamiltonian function $H: T^*M \to \mathbf{R}$ *is defined by*

$$\omega(X_H, Y) = dH(Y).$$

The following proposition from Abraham and Marsden [AbMa 78] characterizes the solution set of a Hamiltonian system.

Proposition 3.4 *In canonical coordinates* (s^i, p_i)

$$X_H = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial s^i}\right) = J.dH$$

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. *A curve* $(\mathbf{s}(t), \mathbf{p}(t))$ *in* T^*M *is an integral of* X_H *iff Hamilton's equation*

$$\dot{s}^i = \frac{\partial H}{\partial p_i}; \dot{p}_i = -\frac{\partial H}{\partial s^i}$$

is satisfied.

Define a Hamiltonian H as follows

$$H: T^*M \to \mathbf{R}: \mathbf{w}_\mathbf{s}^* \overset{H}{\mapsto} sup_{\mathbf{v_s} \in F^0(\mathbf{s})} w_\mathbf{s}^*(\mathbf{v_s}). \tag{3.10}$$

F^0 are closed convex subsets and therefore a supremum is attained. It follows from the definition of the dual cone in equation 3.9 that the function H vanishes for $\mathbf{w}_\mathbf{s}^* \in \delta C^*(\mathbf{s})$ and $\mathbf{v_s} \in \delta C(\mathbf{s})$ for points $\mathbf{s}$ in $M \setminus \mathcal{Z}$ - places where boundary cones are non-empty sets. In general, consider subsets

$$H = constant. \tag{3.11}$$

They define a subset of the cotangent bundle called *Characteristic Set* $\mathcal{C}$ and describes a *scalar partial differential equation.*

A solution of the level surfaces of function H in equation 3.11 is an *integral manifold* described by level sets of a function $z: M \to \mathbf{R}$. For this function to be a solution, the composition map $H \circ dz: M \to \mathbf{R}$ is the required constant map where $dz: M \to T^*M$ is a section of the cotangent bundle, *i.e.*,

$$H \circ dz = constant. \tag{3.12}$$

This is an equation of the *Hamilton-Jacobi* type [AbMa 78, pp. 381].

3.2.5 Forward Projection of a Point

Consider the following interpretation of the Hamilton-Jacobi equation. A section $dz = \sum \frac{\partial z}{\partial s^i} ds^i = \sum p_i ds^i \in T^*M$ represents a normal to the integral manifolds $z = constant$. Recall the definition of function H in equation 3.10. The vanishing of function H on the level surfaces of z therefore implies that the normal to the level surface is orthogonal to the boundary of cone $\delta C(\mathbf{s})$. There are $(n-1)$ non-singular possible normals to any hypersurface in a space of dimension n. One constraint, on the normals of surface given by equation 3.11, specifies $(n-2)$ allowable non-singular normals at the point. This describes a $(n-2)$ family of level surfaces z passing through each point $\mathbf{s} \in M$. The Forward Projection of a point is defined by the envelope of these $(n-2)$ family of surfaces. The following theorem characterizes the *integral manifolds* $z: M \to \mathbf{R}$.

Theorem 3.5 (Hamilton-Jacobi, [AbMa 78]) *Let $X_H: T^*M \to TT^*M$ be the Hamiltonian vector field corresponding to function $H: T^*M \to \mathbf{R}$. Let $\tau_M^*: T^*M \to M$ be the natural projection and $D\tau_M^*: TT^*M \to TM$ be the corresponding derivative map. Let $z: M \to \mathbf{R}$ be a function. Then, the following two statements are equivalent:*

(i) For every curve $\mathbf{s}(t)$ in M satisfying

$$\dot{\mathbf{s}}(t) = D\tau_M^* \circ X_H(dz(\mathbf{s}(t)))$$

the curve $t \to dz(\mathbf{s}(t))$ is an integral of X_H.

(ii) z satisfies the Hamilton-Jacobi equation $H \circ dz = \text{constant}$, i.e.,

$$H(s^i, \frac{\partial z}{\partial s^i}) = \text{constant}.$$

This is the *method of characteristics* used in solving scalar partial differential equations by solving the system of $2n$ ordinary differential equations. The integral curves $(\mathbf{s}(t), \mathbf{p}(t))$ of X_H thus obtained are called *Characteristics.* The elements $\dot{\mathbf{s}}(t) = D\tau_M^* \circ X_H(dz(\mathbf{s}(t)))$ in the tangent space $T_{\mathbf{s}(t)}M$ are called *Characteristic Directions* and the vector field X_H, besides being the *Hamiltonian Vector Field*, is also called the

Characteristic Vector Field (see Figure 3.2). The situation described in the theorem asserts that a *characteristic* remains on the level surfaces of z. Therefore, the forward projection of a point is a surface foliated with $(n-2)$ of such characteristics passing through the point. We assert this situation below [But 82]:

Corollary 3.3 *Let* $H: T^*M \to \mathbf{R}: \mathbf{w}_\mathrm{s}^* \overset{H}{\mapsto} \sup_{\mathbf{v}_\mathrm{s} \in F^0(\mathrm{s})} w_\mathrm{s}^*(\mathbf{v}_\mathrm{s})$ *be a* C^2 *Hamiltonian function corresponding to a differential inclusion problem* $F^0: M \to TM$. *Then, for all points* $\mathbf{s}_0 = \mathbf{s}(0) \in M \setminus \mathcal{Z}$, *the boundary of the forward projection* $\delta\mathrm{FP}(\mathbf{s}_0)$ *is foliated by* $\mathbf{s}(t)$ *for* $0 \leq t \leq t^*$, *where* $(\mathbf{s}(t), \mathbf{p}(t))$ *is an integral curve of* X_H *with initial condition* $(\mathbf{s}(0), \mathbf{p}(0))$ *such that* $H(\mathbf{s}(0), \mathbf{p}(0)) = 0$. *The time* t^* *is infimum of the times* t' *such that* $\mathbf{s}_{\mathbf{q}_1}(t') = \mathbf{s}_{\mathbf{q}_2}(t')$ *for distinct initial conditions* $\mathbf{p}(0) = \mathbf{q}_1$ *and* $\mathbf{p}(0) = \mathbf{q}_2$ *satisfying* $H(\mathbf{s}(0), \mathbf{p}(0)) = 0$. *The inclusion* $\delta\mathrm{FP}(\mathbf{s}_0) \subset \mathbf{s}(t)$ *holds for times* $t > t^*$.

Proof: In canonical coordinates, curve $t \to (s^i(t), p_i(t))$ is an integral curve of X_H. We show that the other condition of (i) of Theorem 3.5 is also satisfied and, by equivalence of (i) and (ii) from theorem 3.5, the curve $t \to (s^i(t), p_i(t))$ keeps H constant. Hence, $t \to s^i(t)$ foliate the boundary of the forward projection.

The Hamiltonian field in local coordinates is given by $X_H = (s^i, p_i, \frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial s^i})$. The map $D\tau_M^*: TT^*M \to TM: (\mathbf{s}, \mathbf{p}, \mathbf{w}_{(\mathbf{s},\mathbf{p})}, \mathbf{a}_{(\mathbf{s},\mathbf{p})}) \overset{D\tau_M^*}{\mapsto} (\mathbf{s}, \mathbf{w}_{(\mathbf{s},\mathbf{p})})$ gives the horizontal element. Therefore, $D\tau_M^* \circ X_H(s^i(t), p_i(t)) = \frac{\partial H}{\partial p_i}$, which is equal to $\dot{s}^i(t)$ since $t \to (s^i(t), p_i(t))$ is an integral curve of X_H.

The initial conditions $(s^i(0), p_i(0))$, such that $H(s^i(0), p_i(0)) = 0$, enumerate all such $s^i(t)$ passing through the point $s^i(0)$. $\diamond$

An alternative proof of the corollary follows from the assertion of Hamiltonian systems that integral curves of X_H keep H constant.

Example 3.1 Consider the following parametrized vector field Y in the Euclidean three space $\mathbf{R}^3$:

$$Y((\theta, \psi), \mathbf{s}) = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} r\cos(\theta)\cos(\psi) \\ r\sin(\theta)\cos(\psi) \\ r\sin(\psi) \end{bmatrix}, \tag{3.13}$$

with $a > r$. The boundary of F^0 is a two sphere parametrized by (θ, ψ) with $\theta, \psi \in [0, 2\pi)$. Assume canonical coordinates (s^i, p_i), on the cotangent space $T^*\mathbf{R}^3$. Consider first a function $F = p_1 r\cos(\theta)\cos(\psi) + p_2 r\sin(\theta)\cos(\psi) + p_3(a + r\sin(\psi))$ which is the inner product of an element (p_1, p_2, p_3) and the elements Y on the boundary of F^0 defined above. Function H is defined as the *supremum* of this function on the domain $\theta, \psi \in [0, 2\pi)$. Setting $\frac{\partial F}{\partial \theta} = 0$ and $\frac{\partial F}{\partial \psi} = 0$, one can obtain $\theta = arctan(p_2/p_1)$ and $\psi = arctan(\frac{p_3}{\sqrt{p_1^2+p_2^2+p_3^2}})$, and hence the function H is given by

$$H = r\sqrt{p_1^2 + p_2^2 + p_3^2} + ap_3. \tag{3.14}$$

The *characteristic vector field* of function H is given by

$$\begin{aligned}
\dot{s}^i &= \frac{\partial H}{\partial p_i} = \frac{rp_i}{\sqrt{p_1^2 + p_2^2 + p_3^2}}, i = 1, 2, \\
\dot{s}^3 &= \frac{\partial H}{\partial p_3} = \frac{rp_3}{\sqrt{p_1^2 + p_2^2 + p_3^2}} + a, \\
\dot{p}_i &= \frac{\partial H}{\partial s^i} = 0, i = 1, 2, 3.
\end{aligned}$$

Integrating first $\dot{p}_i = 0$, and then substituting the resulting constants in the rest, one can obtain the following *characteristics*

$$\begin{aligned}
p_i(t) &= c_i, i = 1, 2, 3; \\
s^i(t) - s^i(0) &= \frac{rc_i}{\sqrt{c_1^2 + c_2^2 + c_3^2}} t, i = 1, 2, \\
s^3(t) - s^3(0) &= \frac{rc_3}{\sqrt{c_1^2 + c_2^2 + c_3^2}} t + at,
\end{aligned} \tag{3.15}$$

with constants c_i. The vector $(s^1(0), s^2(0), s^3(0), p_1(0) = c_1, p_2(0) = c_2, p_3(0) = c_3) \in T^*\mathbf{R}^3$ satisfies function H in equation 3.14. In this example, with the initial condition that function H vanishes, it follows that $\frac{r}{a} = -\frac{c_3}{\sqrt{c_1^2+c_2^2+c_3^2}}$, and it is possible to eliminate constants c_i and t and obtain a canonical cone equation

$$z = \frac{(s^1(t) - s^1(0))^2}{r^2} + \frac{(s^2(t) - s^2(0))^2}{r^2} - \frac{(s^3(t) - s^3(0))^2}{a^2 - r^2} = 0$$

with vertex $(s^1(0), s^2(0), s^3(0))$ and axis aligned with the s^3 axis. $\diamond$

As pointed out earlier, function H attains zero as supremum in $M \setminus \mathcal{Z}$. At such points the cone boundary field $\delta C(\mathrm{s})$ are tangential to the level surfaces of function z.

3.2.6 Forward and Back Projection of a Regular Closed Set

Consider a closed non-null regular set $\mathcal{I}$. The boundary of the forward projection $\delta\mathrm{FP}(\mathcal{I})$ can be arbitrarily complex. It is possible that solutions leave the closed set $\mathcal{I}$ at all points of the boundary and, therefore, locally, the boundary of the forward projection is null. On the other hand, the forward projection of a closed set can be the set itself. In particular, if all solution sets for all points in $\mathcal{I}$ remain in $\mathcal{I}$, then they make the set $\mathcal{I}$ strongly invariant. Aubin and Cellina [AubCel 84, pp. 233] give conditions for a closed convex set to be so invariant. In such a case, the boundary of the forward projection is the boundary $\delta\mathcal{I}$ of the closed set $\mathcal{I}$. In contrast with these two extremes, we identify an intermediate case in this section when the boundary of the forward projection of $\mathcal{I}$ consists of subset of its own boundary $\delta\mathcal{I}$ and the integral manifolds.

Assume that the topological boundary $\delta\mathcal{I}$ of the closed set $\mathcal{I}$ is a level hypersurface of a function $f_{\mathcal{I}}: M \to \mathbf{R}$. The boundary of the forward projection of this closed set $\mathcal{I}$ consists of parts of its own boundary $\delta\mathcal{I}$ and the *integral manifolds* of the form $z: M \to \mathbf{R}$ satisfying the Hamilton-Jacobi equation 3.12. Such integral manifolds are tangential to the boundary $\delta\mathcal{I}$. Let Λ denote a submanifold on the boundary $\delta\mathcal{I}$ where an integral manifold is tangential to the boundary. Let the natural inclusion of this subset in M be $\imath: \Lambda \to M$. The submanifold Λ is called an *Initial Manifold.* The integral manifold z is a solution of an *Initial Value Problem* with *initial manifold* Λ of the *Hamilton-Jacobi* equation 3.12. The conditions for the existence and uniqueness of an integral manifold tangential to the boundary $\delta\mathcal{I}$ are those that are necessary for a well defined forward projection of the closed set $\mathcal{I}$.

Consider the *characteristic set* $C \subset T^*M$ of the Hamiltonian function H in equation 3.11. By definition, there exists an *integral manifold* of H if and only if the initial manifold Λ is contained in the characteristic set. In addition, if the integral manifold is a submanifold, then the initial manifold must also satisfy a genericity condition.

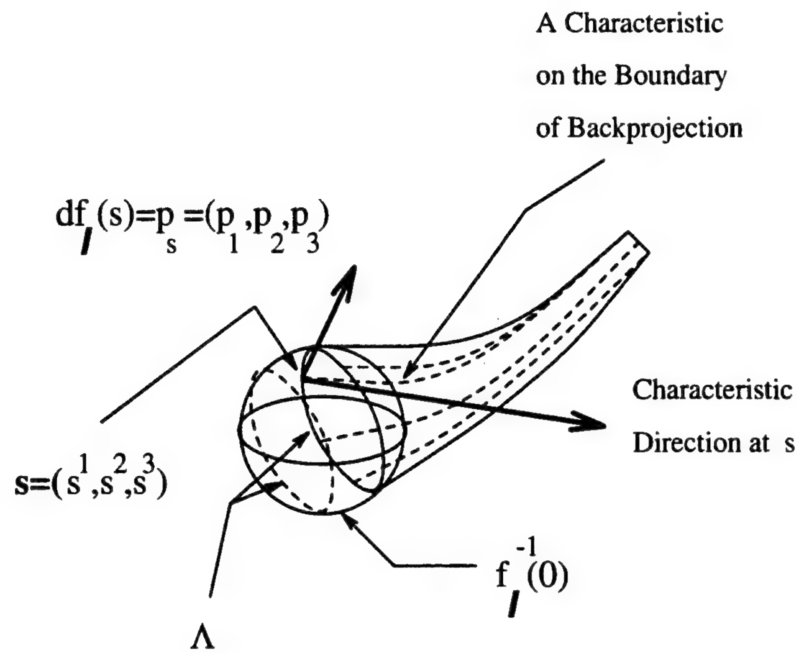


Figure 3.2: Boundary of Backprojection of a regular, closed set $\{\mathbf{s} | f_{\mathcal{I}}(\mathbf{s}) \leq 0\}$

Next we define a relation between the tangent space of the initial manifold and the characteristic direction:

Definition 3.3 *A point* $\mathbf{s}$ *in the initial manifold* Λ*, a subset of the characteristic set* $\mathcal{C}$*, is a* non-characteristic point *if the* characteristic direction $D\tau_M^* \circ X_H(dz(c(t)))$ *is not contained in the tangent space* $D\imath(T_{\mathbf{s}}\Lambda)$.

An *initial manifold* is *non-characteristic* if all its points are *non-characteristic.* The following theorem [Arn 88] gives the existence and uniqueness conditions of an integral manifold through an initial manifold.

Theorem 3.6 *Let* $\mathbf{s}$ *be a* non-characteristic point *of the initial manifold* Λ*. There exists a neighborhood* U *of* $\mathbf{s}$ *such that the integral manifold in* $\mathcal{C} \cap U$ *with the initial manifold* $\Lambda \cap U$ *exists and is locally unique. The integral manifold consists of characteristics passing through points in the initial manifold* Λ.

The Theorem 1.1 gives condition for the existence and uniqueness of a well-defined initial manifold Λ which are also conditions necessary for the existence of an integral manifold that is locally $\delta\mathrm{FP}(\mathcal{I})$, the boundary of the forward projection.

Proof of Theorem 1.1: An application of the *Preimage Theorem* shows that Λ is a well-defined $(n-2)$ submanifold. For Λ to be a subset of $\mathcal{C}$, note that $H \circ df_{\mathcal{I}}|_{\Lambda}$ vanishes identically.◇

The tangentiality of the boundary of set $\mathcal{I}$ and the integral manifold $\delta \mathrm{FP}(\mathcal{I})$ at points of Λ can occur in two ways. The hyperplane along which the integral manifold and the boundary of the closed set $\delta\mathcal{I}$ are tangential can keep the set F^0 and the set $\mathcal{I}$ on opposite sides or on the same side. We assume here that an appropriate distinction is made in choosing a initial manifold Λ_i, a subset of Λ, that keeps the two on the same side for the forward projection and on opposite sides with negative Hamiltonian H for backprojection. An explicit check of this part is also done to characterize the *forward projection of singular invariant set* in Section 3.2.10.

Example 3.2 Consider characterizing the initial manifold Λ for the backprojection of a sphere for the differential inclusion problem of Example 3.1. Let the boundary of initial set $\mathcal{I}$ (in accordance with the notation in Chapter 2, this should be denoted as $\mathcal{G}$ for goal, since the backprojection is that of a goal and not an initial set $\mathcal{I}$) be given by the zero set of the function $f_{\mathcal{I}} = (s^1)^2+(s^2)^2+(s^3)^2-b^2$. The corresponding section $df_{\mathcal{I}}$ is $\sum_{i=1}^{3} 2(s^i)d(s^i)$. The composition map $H \circ df_{\mathcal{I}}$ is $-2r\sqrt{(s^1)^2+(s^2)^2+(s^3)^2} - 2as^3$, where H here is the negative of the H in equation 3.14. Map $f_{\mathcal{I}} \times H \circ df_{\mathcal{I}}$ has zero as a regular value and, therefore, $\Lambda = (f_{\mathcal{I}} \times H \circ df_{\mathcal{I}})^{-1}(0) \equiv \{(s^1)^2+(s^2)^2+(s^3)^2-b^2=0\} \cap \{s^3 = \pm\frac{br}{a}\}$. Of the two connected components, the component $\Lambda_1 \equiv \{(s^1)^2+(s^2)^2+(s^3)^2-b^2=0\} \cap \{s^3 = \frac{br}{a}\}$ keeps the set $\mathcal{I}$ and $-F^0$ from equation 3.13 on the opposite sides. The integral manifold of $-H$ with H from equation 3.14 with the initial manifold Λ_1 defines the boundary of backprojection locally.◇

3.2.7 Cone Boundary Vector Field for Smooth C^0 Neighborhoods

In this section, we give an explicit computation of the cone boundary field from a smooth parametrization of the neighborhood boundary δF^0 in equation 3.4. We also show that, with some assumptions, the cone boundary fields must be degenerate at

the boundary of singular invariant sets.

First, we define the *vertical lift* and *horizontal part* of an element of the second tangent bundle [AbMa 78]. Consider elements $(\mathbf{s}, \mathbf{v_s})$ and $(\mathbf{s}, \mathbf{w_s})$ in $T_\mathbf{s}M$. The *vertical lift* of $(\mathbf{s}, \mathbf{w_s})$ relative to $(\mathbf{s}, \mathbf{v_s})$ is

$$(\mathbf{w_s})^l_{\mathbf{v_s}} = \frac{d}{dt}(\mathbf{v_s} + t\mathbf{w_s})|_{t=0} \in T_{\mathbf{v_s}}(TM).$$

The *horizontal part* of a vector $\beta_{\mathbf{v_s}} \in T_{\mathbf{v_s}}TM$ is $D\tau(\beta_{\mathbf{v_s}})$, where $\tau = (TTM, TM, \tau)$, is the second tangent bundle. In natural charts, it is easy to see that $(\mathbf{w_s})^l_{\mathbf{v_s}} = ((\mathbf{s}, \mathbf{v_s}), 0, \mathbf{w_s})$ and the horizontal part of an element $\mathbf{a}_{\mathbf{v_s}} = ((\mathbf{s}, \mathbf{v_s}), \mathbf{w}_{\mathbf{v_s}}, \mathbf{a}_{\mathbf{v_s}})$ is $(\mathbf{s}, \mathbf{w}_{\mathbf{v_s}})$. An element in TTM is a vertical lift of an element in TM if, and only if, the horizontal part is zero.

Consider the case when the boundary of F^0 is parametrized as a smooth map Y in equation 3.4. Consider for each $\mathbf{s} \in M$, the map

$$g_\mathbf{s}: T_{S^{n-1}\times M}(S^{n-1}) \to \mathbf{R}^n: (\mathbf{p}, \mathbf{v_p}) \overset{g_\mathbf{s}}{\mapsto} \pi_4 \circ (D_1Y(\mathbf{p}, \mathbf{s})\mathbf{v_p} + (Y(\mathbf{p}, \mathbf{s}))^l_{Y(\mathbf{p},\mathbf{s})}), \qquad (3.16)$$

where $\pi_4(\mathbf{s}, \mathbf{v_s}, \mathbf{w}_{(\mathbf{s},\mathbf{v_s})}, \mathbf{a}_{(\mathbf{s},\mathbf{v_s})}) = \mathbf{a}_{(\mathbf{s},\mathbf{v_s})}$ is the natural projection on the fourth factor. If sphere S^{n-1} is mapped diffeomorphically by map Y, then the tangent space to the image sphere at any point $Y(\mathbf{p}, \mathbf{s})$ is an $(n-1)$-dimensional hyperplane. We look for points $(\mathbf{p}, \mathbf{v_p}) \in TS^{n-1}$ whose image by the map $g_\mathbf{s}$ aligns with the element $Y(\mathbf{p}, \mathbf{s})$. These are points where map $g_\mathbf{s}$ vanishes for some $\mathbf{v_p}$. The corresponding elements $Y(\mathbf{p}, \mathbf{s})$ are tangential to the image sphere and lie on the boundary cone $\delta C(\mathbf{s})$. Notice that at points $\mathbf{s}$ in $\mathring{\mathcal{Z}}$ which are interior points of singular invariant sets, map $g_\mathbf{s}$ does not vanish because no element $Y(\mathbf{p}, \mathbf{s})$ lies in the tangent space to the image sphere.

If Y maps $(n-1)$-sphere diffeomorphically at each point so that D_1Y is onto, and the map $g_\mathbf{s}$ is surjective, then $g_\mathbf{s}^{-1}(0)$ defines a smooth $(n-2)$-dimensional subset of an $(n-1)$-sphere at each point in $M \setminus \mathring{\mathcal{Z}}$. To see this, consider the derivative map by $D(g_\mathbf{s}) = [D_{11}Y\mathbf{v_p} + D_1YD_1\mathbf{v_p} + D_1Y \quad D_1Y]$ which has at least rank n, since $g_\mathbf{s}$ is surjective. The linear map D_1Y is surjective, so it has rank $(n-1)$. Therefore, elements $\mathbf{v_p}$ are uniquely determined. The remaining one condition defines a codimension one set on the sphere S^{n-1}. The subset of codimension one of S^{n-1} is an $(n-2)$-sphere for points in $M \setminus \mathring{\mathcal{Z}}$, since the multivalued map F^0 is convex.

Let f be the implicitly defined function that takes points in S^{n-2} to S^{n-1} so that the cone boundary vector field Y_B is defined as

$$Y_B(\mathbf{r},\mathbf{s}) = Y(f(\mathbf{r},\mathbf{s}),\mathbf{s}) \tag{3.17}$$

satisfying $g_{\mathbf{s}}^{-1}(0)$ from equation 3.16, *i.e.*, $\pi_4 \circ D_1 Y(f(\mathbf{r},\mathbf{s}),\mathbf{s})\mathbf{v}_{f(\mathbf{r},\mathbf{s})} + Y(f(\mathbf{r},\mathbf{s}),\mathbf{s}) = 0$. More precisely, map f is defined as $f\colon S^{n-2} \times M \backslash \overset{\circ}{\mathcal{Z}} \to S^{n-1}$.

Example 3.3 Consider a parametrized vector field Y in plane given by

$$Y(p,(q_1,q_2)) = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} r\cos(p) \\ r\sin(p) \end{bmatrix},$$

where $p \in (0, 2\pi]$ and r is a scalar denoting the radius of the control uncertainty circle. The corresponding cone boundary vector field Y_B consists of a left and right component corresponding to two points $\{p_0, p_1\} \in S^0$ defined on $\mathbf{R}^2 \backslash \{c^2q_1^2 + d^2q_2^2 - r^2 \leq 0\}$. They are

$$Y_B(\{p_0,p_1\},(q_1,q_2)) = \begin{bmatrix} \frac{-(cr^2q_1) + c^3q_1^3 + cd^2q_1q_2^2 \mp drq_2\sqrt{-r^2+c^2q_1^2+d^2q_2^2}}{c^2q_1^2+d^2q_2^2} \\ \frac{-(dr^2q_2) + c^2dq_1^2q_2 + d^3q_2^3 \pm crq_1\sqrt{-r^2+c^2q_1^2+d^2q_2^2}}{c^2q_1^2+d^2q_2^2} \end{bmatrix},$$

where $\pm$ corresponds to two points $\{p_0, p_1\} \in S^0$. The function $f\colon S^0 \times \mathbf{R}^2 \to S^1$ defined by $p = arctan(-drq_2 \pm cq_1\sqrt{-r^2 + c^2q_1^2 + d^2q_2^2}, -crq_1 \mp dq_2\sqrt{-r^2 + c^2q_1^2 + d^2q_2^2}) \in (0, 2\pi]$ can be extended to the boundary $\delta\mathcal{Z}_1 \equiv \{c^2q_1^2 + d^2q_2^2 - r^2 = 0\}$ of the singular set. However, cone boundary vector field Y_B at the boundary of the singular set is not differentiable in this case, neither is map f. $\diamond$

If a cone boundary vector field Y_B is defined which is continuously differentiable on the boundary of the singular invariant set and the boundary is a diffeomorphic image of S^{n-1}, then the following proposition shows that the field Y_B can not be of full rank at points of singularity.

Proposition 3.5 *Consider a C^1 cone boundary map Y_B given by equation 3.17 on $M\backslash(\mathcal{Z}\backslash\delta\mathcal{Z}_i)$, where D_1Y is onto $\delta\mathcal{Z}_i$. If Y_B for a point $\mathbf{r}_0 \in S^{n-2}$ has a singularity in $\delta\mathcal{Z}_i$, then the linear map $(DY_B)_{\mathbf{r}_0}$ at the singular point is degenerate for* $\dim(M) > 1$.

Proof: The singularity of the cone boundary vector field Y_B are subsets of the singularity of the field Y. A construction of the set of singularities of field Y, denoted $\delta\mathcal{Z}_i$, is given in the proof of Proposition 3.1. First consider field $(Y_B)_{\mathbf{r}_0}$ for a point $\mathbf{r}_0 \in S^{n-2}$ and a corresponding isolated singularity $\mathbf{s}_0 \in \delta\mathcal{Z}_i$, the i^{th} component of the set of singularities of the field Y, so that $(Y_B)_{\mathbf{r}_0}(\mathbf{s}_0) = Y_B(\mathbf{r}_0, \mathbf{s}_0) = 0$. There exists a point $\mathbf{p}_0 \in S^{n-1}$ so that $Y(\mathbf{p}_0, \mathbf{s}_0) = 0$. Consider map $g_i\colon S^{n-1} \to M$ constructed in the proof of Proposition 3.1 that maps the $(n-1)$-sphere onto the set of singular points of the field Y. In particular, $g_i(\mathbf{p}_0) = \mathbf{s}_0$. If field Y_B is C^1 on $\delta\mathcal{Z}_i$, the map f is also well defined and differentiable. Differentiating equation 3.17, we get

$$D_2 Y_B = D_2 Y + D_1 Y \, D_2 f. \tag{3.18}$$

The identity $f(\mathbf{r}_0, g_i(\mathbf{p}_0)) = \mathbf{p}_0$ follows. Differentiating this with respect to points in the $(n-1)$ sphere we get $D_2 f \circ Dg_i = id_{n-1\times n-1}$. Multiplying both sides of equation 3.18 by Dg_i and using the derivative relationship $D_2 Y Dg_i = -D_1 Y$ on $\delta\mathcal{Z}_i$ from the Proposition 3.1, we establish

$$\begin{aligned} D_2 Y_B \, Dg_i &= D_2 Y \, Dg_i + D_1 Y \, D_2 f \, Dg_i \\ &= -D_1 Y + D_1 Y \, id_{n-1\times n-1} \\ &= 0_{n-1\times n-1}. \end{aligned}$$

If $D_1 Y|_{\delta\mathcal{Z}_i}$ is onto, Dg_i is of rank $(n-1)$. Therefore, $D_2 Y_B$ can not be of full rank for $n > 1$. $\diamond$

3.2.8 Convex Polytope as C^0 Boundary

When multivalued map F^0 is smooth, Proposition 3.5 shows that the cone boundary vector field Y_B at the singular points must be degenerate. Consider *regular*, *convex polytopes* as the boundary of F^0 at all points. Then, there exist vector fields whose cone boundary field has non-degenerate isolated zeroes. In these cases, the forward projection of a singular invariant subset is well defined. This and the following sections give constructions related to a convex polytope map F^0. A subsequent section considers the forward projection of singular invariant subsets.

Consider the following representation of F^0 as polytopes. A polytope is bounded by hyperplanes $\{\mathcal{P}_i, i = 1, r\}$ in the tangent space. A specification of a hyperplane $\mathcal{P}_i$ is a pair $\{X_i, \alpha_i\}$, where $X_i\colon M \to TM$ is a vector field on M, and $\alpha_i\colon M \to T^*M$ is a section of the cotangent bundle. A hyperplane $\mathcal{P}_i$ is an $(n-1)$-dimensional subspace defined by $ker(\alpha_i)$, the kernel of the one-form, displaced by the vector field X_i. If $\{X_i, \alpha_i\}$ specifies $\mathcal{P}_i$, then $\{X_i, -\alpha_i\}$ is another possible specification. Between the two choices, let the positive half-spaces of the hyperplanes define the polytope.

Definition 3.4 *A polytope neighborhood is*

$$\{X(s) \in F^0(s) | \alpha_i(s)(X(s)) \geq \alpha_i(s)(X_i(s)), \forall i = 1, r\}, \tag{3.19}$$

where α_i's are non-degenerate, *any two of them are linearly independent, and together they define a* regular, convex *polytope at all points of M.*

The intersection of the positive half spaces of a set of hyperplanes is denoted $\{\mathcal{P}_i, i = i_1 \dots i_m\}$ or more explicitly $\{\{X_i, \alpha_i\}, i = i_1 \dots i_m\}$. In this notation $F^0 \equiv \{\{X_i, \alpha_i\}, i = 1 \dots r\}$. An intersection $\cap_{k=i_1 i_2 \dots i_m} \mathcal{P}_k$ of m hyperplanes is denoted $\mathcal{P}_{i_1 i_2 \dots i_m}$. A *Supporting Hyperplane* $\{X, \alpha\}$ to a convex set K satisfies

$$\alpha(X) = \min_{Y \in K} \alpha(Y) \tag{3.20}$$

[Lang 72]. A *Supporting Hyperspace* to a convex set K is a tuple $\{0, \alpha\}$ satisfying the previous relationship in equation 3.20.

Consider two hyperplanes $\mathcal{P}_i = \{X_i, \alpha_i\}$ and $\mathcal{P}_j = \{X_j, \alpha_j\}$ so that $\mathcal{P}_{ij}$ is a face of the polytope F^0. A hyperspace $\mathcal{P}$ containing segment $\mathcal{P}_{ij}$ is given by

$$\mathcal{P} = \{X = 0, \alpha = \alpha_i(X_i)\alpha_j - \alpha_j(X_j)\alpha_i\} \tag{3.21}$$

(see Figure 1.1). The hyperspace $ker(\alpha)$ specifies an $(n-1)$-dimensional distribution. Let this $(n-1)$-distribution determined by the hyperspaces $\mathcal{P}_i$ and $\mathcal{P}_j$ be *integrable* in the sense of *Frobenius* [Spiv 79], *i.e.,*

$$\alpha \wedge d\alpha \equiv 0.$$

Consider a C^1 vector field X that belongs to this hyperspace, *i.e.,* $\alpha(X) \equiv 0$ and $X \in \mathcal{P}_{ij}$. Let X be non-singular on an open set $U \subset M$, and $\phi(t, s)$ be the corresponding flow. Consider the following proposition from Arnold [Arn 88]:

Proposition 3.6 *Let $X: U \rightarrow TM$ and $\alpha: U \rightarrow T^*M$ be smooth non-singular contravariant and covariant sections such that $\alpha(X) \equiv 0$. If an $(n-1)$-dimensional distribution determined by* $\ker(\alpha)$ *is integrable, then the flow ϕ of X keeps the distribution determined by α invariant, i.e.,*

$$(\phi_t)_*(\mathbf{s})(\ker(\alpha(\mathbf{s}))) = \ker(\alpha(\phi_t(\mathbf{s}))).$$

Consider $\delta\mathcal{Z}_j$, the boundary of a singular invariant subset $\mathcal{Z}_j$. A segment on this boundary is the image of a hyperplane $\mathcal{P}_i$. A point $\mathbf{s}_0$ on this segment is the image of a point $\mathbf{p}_0 \in S^{n-1}$, so that the boundary vector field $Y(\mathbf{p}_0, \mathbf{s}_0)$ of equation 3.4 vanishes. It follows from equation 3.19 that $\alpha_i(\mathbf{s}_0)(X_i(\mathbf{s}_0)) = 0$. The subset of M, where functions $\alpha_i(X_i): M \rightarrow \mathbf{R}$ for a hyperplane $\mathcal{P}_i$ vanishes, is defined as a *transition hypersurface* denoted $\mathcal{T}_i$.

Proposition 3.7 *The transition hypersurfaces $\mathcal{T}_i = \alpha_i(X_i)^{-1}(0)$ are smooth submanifolds of M if the boundary field Y is a transversal C^1 section.*

Proof: We show that zero is a regular value of the function $\alpha_i(X_i)$ if field Y is a transversal section. Then, the set $(\alpha_i(X_i))^{-1}(0)$ is a smooth submanifold of M by the Preimage Theorem [GP 74]. Consider a chart $(U, \mathbf{x})$ of M such that $Y(\mathbf{p}_0, \mathbf{s}_0) = 0 \in \mathcal{P}_i$, and $\mathbf{s}_0 \in U$. With the representation of a vector field Y as $Y_{\mathbf{p}_0}(\mathbf{s}) = a^1(\mathbf{s})\frac{\partial}{\partial x^1} + a^2(\mathbf{s})\frac{\partial}{\partial x^2} + \ldots + a^n(\mathbf{s})\frac{\partial}{\partial x^n}$ and an element $\alpha \in T^*M$ in the dual coordinate system as $\alpha(p) = \alpha_1 dx^1 + \alpha_2 dx^2 + \ldots + \alpha_n dx^n$, a function $\alpha(Y): M \rightarrow \mathbf{R}^n$ is given by

$$\alpha(Y) = a^1(\mathbf{s})\alpha_1(\mathbf{s}) + a^2(\mathbf{s})\alpha_2(\mathbf{s}) + \ldots + a^n(\mathbf{s})\alpha_n(\mathbf{s}).$$

The derivative of this function is

$$\begin{aligned} D(\alpha(Y)) &= \sum_j \left(\sum_i \frac{\partial a^i(\mathbf{s})}{\partial x^j}\alpha_i(\mathbf{s}) + \sum_i a^i(\mathbf{s})\frac{\partial \alpha_i(\mathbf{s})}{\partial x^j} dx^i \right) \\ &= [\alpha_i] \left[\frac{\partial a^i(\mathbf{s})}{\partial x^j} \right] + [a_i] \left[\frac{\partial \alpha^i(\mathbf{s})}{\partial x^j} \right] \\ &= [\alpha_i] DY + [Y]^T D(\alpha), \end{aligned}$$

where $[\alpha_i] = [\alpha_1\ \alpha_2 \ldots \alpha_n]$ is a row vector and $Y = [a^1\ a^2 \ldots a^n]^T$ is a column vector. Differentiating the equality form of constraint in equation 3.19,

$$\begin{aligned} & [\alpha_i]\,[DY] + Y^T D(\alpha_i) - D(\alpha_i(X_i)) = 0, \\ \Rightarrow\ & [\alpha_i]\,[DY] = D(\alpha_i(X_i)) - Y^T D(\alpha_i) \end{aligned} \tag{3.22}$$

By assumption the α_i's are non-degenerate. It follows from equation 3.19, that Y is singular when $\alpha_i(X_i)$ vanishes. So, the second term on the right hand side vanishes. DY at this point is full rank because Y is transversal. Therefore, $D(\alpha_i(X_i))$ is surjective.◇

A hyperspace $\mathcal{P}$ of the form in equation 3.21 is supporting to F^0 at those points $\mathrm{s} \in M$ where

$$\alpha_i(\mathrm{s})(X_i(\mathrm{s})) \geq 0, \quad \textit{and} \quad \alpha_j(\mathrm{s})(X_j(\mathrm{s})) \leq 0 \tag{3.23}$$

(see Proposition 3.10 later). We call the submanifolds $(\alpha_i(X_i))^{-1}(0)$ a *transition hypersurfaces* $\mathcal{T}_i$ because a supporting hyperspace $\mathcal{P}$ can cease to be supporting at points of these submanifold.

Consider points in $M \setminus \mathcal{Z}$ where neighborhood F^0 does not contain the origin, *i.e.*, $0 \notin F^0$. The elements of the cone boundary vector field Y_B at any point s are those that belong to a supporting hyperspace of the convex polytope $F^0(\mathrm{s})$. Consider a region of M where a hyperspace $\mathcal{P}$ of the form in equation 3.21 is supporting. In this region, consider a C^1 field Y_B so that $Y_B \in \mathcal{P}_{ij}$. By Proposition 3.6, flow ϕ_B of Y_B keeps the hyperplane $ker(\alpha)$ invariant. As a result, flow ϕ_B is a leaf on the boundary of the forward projection of a point in this region. All such flows ϕ_B of vector fields Y_B on the face $\mathcal{P}_{ij}$ form a smooth surface of the boundary of the forward projection. Flows belonging to all such supporting hyperplanes foliate the complete boundary δFP.

3.2.9 A Piecewise Smooth Singular Invariant Set

The boundary $\delta\mathcal{Z}_j$ of any singular invariant set is a union of segments of transition hypersurfaces. Any two hypersurfaces $\mathcal{T}_k$ and $\mathcal{T}_l$ intersect to define a segment $\mathcal{T}_{kl}$ of codimension two on the boundary $\delta\mathcal{Z}_j$, because the two α_i's are, by assumption, linearly independent. Three of these hypersurfaces define a codimension three segment. The boundary $\delta\mathcal{Z}_j$ consists of a disjoint union of segments of codimension one to n - a codimension n segment being a vertex defined by the intersection of n or more such hypersurfaces. Consider the facial structure of a singular invariant set $\mathcal{Z}_j$. Call faces $\mathcal{T}^j_{i_1 \ldots i_m}, m \geq 1$, the *proper faces.* Add to them $\mathring{\mathcal{Z}}_j$, the interior of the singular

invariant subset and $\emptyset$, the null set, which are *improper faces* of dimensions n and -1 respectively. The singular invariant set $\mathcal{Z}_j$ is thus a disjoint union of faces of dimension -1 to n. A *proper face* $\mathcal{T}^j_{i_1 i_2 \ldots i_m}$, is formed by the intersection of hypersurfaces $\{\mathcal{T}_{i_p}, p = 1, m\}$. Each face $\mathcal{T}^j_{i_1 \ldots i_m}$ is differentiable. The boundary $\delta\mathcal{Z}_j$ is union of the proper faces and is piecewise differentiable. Notice that the symbol for a transition hypersurface is $\mathcal{T}_i$, and that of a its subset on the boundary of a singular invariant set $\delta\mathcal{Z}_j$ is $\mathcal{T}^j_i$.

Lemma 3.1 (Orientation Lemma) *An orientation of $D(\alpha_i(X_i))$ is determined at all points of proper faces $\mathcal{T}^j_i \subset \mathcal{Z}_j$ by the choice of the orientation of α_i in equation 3.19.*

Proof: Consider a point $\mathbf{s} \in \mathcal{T}^j_i$ and a neighborhood $B_r(\mathbf{s}) \subset M$ so small that it does not intersect any other transition hypersurfaces. The neighborhood is split in two halves by the surface $\mathcal{T}^j_i$, one of which is $B_r(\mathbf{s}) \cap \mathring{\mathcal{Z}}_j$. For $\mathbf{s}'$ in this half, the origin lies in the positive half space of the hyperplane $\mathcal{P}_i$ by virtue of equations 3.19 and 3.1. Therefore, $\alpha_i(\mathbf{s}')(0) = 0 > \alpha_i(\mathbf{s}')(X_i(\mathbf{s}'))$. Similarly, for points $\mathbf{s}'$ in the other half $\alpha_i(\mathbf{s}')(X_i(\mathbf{s}')) > 0$. Hence, an orientation of $D(\alpha_i(X_i))$ at $\mathbf{s} \in \mathcal{T}^j_i$ is fixed. There exists a $\delta < 0$ so that $\alpha_i(X_i)^{-1}(\delta) \cap \mathcal{Z}_j \neq \emptyset$, *i.e.*, negative value level hypersurfaces of $\alpha_i(X_i)$ belong to the interior of the singular invariant set. Intuitively, this orients $D(\alpha_i(X_i))$ seen as a normal to $\mathcal{T}^j_i$, in dual coordinates, pointing from the inside of the singular invariant set $\mathcal{Z}_j$ to the outside. $\diamond$

Consider now the tangents to the proper faces of the singular invariant subsets. The tangent space to a transition hypersurface $\mathcal{T}_i$ is $ker(D(\alpha_i(X_i)))$ which is also a tangent space of $\mathcal{T}^j_i$, the subset of $\delta\mathcal{Z}_j$. For brevity, we denote $D(\alpha_i(X_i))$ by $D\mathcal{P}_i$ so that $D\mathcal{P}_i : M \to T^*M$ is a section of the cotangent bundle. The tangent spaces $T\mathcal{T}^j_{i_1 \ldots i_m}$ of proper faces of codimension $m \geq 1$ are defined analogously. They are $\bigcap_{k=i_1 \ldots i_m} ker(D\mathcal{P}_k)$, the intersection of the tangent spaces of each participating face. Also, the tangent spaces of the participating faces bound a convex set as shown below.

Proposition 3.8 *The hyperplanes $\{0, D\mathcal{P}_k\}, k = i_1 \ldots i_m$, bound a convex set at a point $\mathbf{s} \in \mathcal{T}^j_{i_1 i_2 \ldots i_m}$ if hyperplanes $\mathcal{P}_k = \{X_k, \alpha_k\}, k = i_1, \ldots i_m$, bound a convex set.*

Proof: At points $\mathbf{s} \in \mathcal{T}^j_{i_1 i_2 \ldots i_m}$, recall that equation 3.22 reduces to $[\alpha_i][DY] = D(\alpha_i(X_i))$. Map DY is linear, and a convex set is mapped by a linear map to a convex set [Lang 72]. ◇

In addition to tangents, we define *Supporting Hyperspaces* and *Supporting Subspaces* at a point $\mathbf{s}$ of a proper face $\mathcal{T}^j_{i_1 \ldots i_m}$. First recall that if $\{\{0, D\mathcal{P}_k\}, k = i_1 \ldots i_m\}$ is a convex set, then so is $\{\{0, -D\mathcal{P}_k\}, k = i_1 \ldots i_m\}$.

Definition 3.5 *A hyperspace*

$$\{\ker(\alpha), \alpha \in T^*_{\mathbf{s}}M | \mathbf{v_s} \in T_{\mathbf{s}}M, D\mathcal{P}_k(\mathbf{s})(\mathbf{v_s}) \leq 0, \forall k \in i_1 \ldots i_m \Rightarrow \alpha(\mathbf{v_s}) \leq 0\} \quad (3.24)$$

is a Supporting Hyperspace *to the set $\mathcal{Z}_j$ at a point $\mathbf{s} \in \mathcal{T}^j_{i_1 \ldots i_m}$. Denote all such supporting hyperspaces by $\mathcal{H}(\mathcal{T}^j_{i_1 \ldots i_m}, \mathbf{s})$. A* Strictly Supporting Hyperspace *is a supporting hyperspace distinct from every $D\mathcal{P}_k, k = i_1, \ldots i_m$.*

The supporting hyperspace at a point in $\mathcal{T}^j_i$ is uniquely defined by the kernel of $D\mathcal{P}_i$. However, there exists a family of hyperspaces at points on segments of codimension two or more. The following proposition shows all such supporting hyperspaces.

Proposition 3.9 *All* Supporting Hyperspace *of a convex set defined by hyperplanes $\{X, \beta_k\}, k = i_1, \ldots, i_m$, are of the form $\{X, \sum_{k=i_1,\ldots,i_m} c_k \beta_k\}, c_k \geq 0$.*

Proof: Recall that if $\{X, \alpha\}$ is a supporting hyperplane to a convex set K, then $\alpha(X) = \min_{Y \in K} \alpha(Y)$. The proof is by contradiction. Without loss of generality, choose $X = 0$. For any point $\mathbf{v}$ in the convex set, $\beta_k(\mathbf{v}) \geq 0$ and $\alpha(\mathbf{v}) \geq 0$, where $\{0, \alpha\}$ is a supporting hyperspace. First, a supporting hyperspace $\{0, \alpha\}$ contains the intersection of all the kernels of the defining hyperspaces β_k. Therefore, α is linear combination of these hyperspaces, *i.e.*, $\alpha = \sum_{k=i_1,\ldots,i_m} c_k \beta_k$. If any $c_j < 0$, then consider a point $\mathbf{v}$ in the convex set so that $\beta_k(\mathbf{v}) = 0, k \neq j$ and $\beta_j(\mathbf{v}) > 0$. Therefore, $\alpha(\mathbf{v}) = \sum_{k=i_1,\ldots,i_m} c_k \beta_k(\mathbf{v}) = c_j \beta_j(\mathbf{v}) < 0$ by construction. Hence, $\{0, \alpha\}$ is not a supporting hyperspace.◇

Definition 3.6 *A subspace $W(\mathbf{s})$ is a* supporting subspace *if it is a subspace of a supporting hyperspace $\mathcal{P}$ in $\mathcal{H}(\mathcal{T}^j_{i_1 \ldots i_m}, \mathbf{s})$. A* Strictly Supporting Subspace $W(\mathbf{s})$ *is a subspace of a strictly supporting hyperspace such that $W(\mathbf{s}) \cap T\mathcal{T}^j_{i_1 \ldots i_m} = 0$, the trivial element.*

Consider a local neighborhood $B_r(\mathrm{s})$ of a point $\mathrm{s} \in \mathcal{T}^j_{i_1 \ldots i_m}$, where $B_r(\mathrm{s})$ is an open ball of radius $r > 0$. Let r be arbitrarily small so that each transition hypersurface $\mathcal{T}_{i_k}, k = 1 \ldots m$, intersects the boundary of the ball δB_r transversally and the intersection of the ball with other transition hypersurfaces is empty. Such a neighborhood exists because transition hypersurfaces intersect transversally. Each transition hypersurface $\mathcal{T}_{i_k}$ divides the ball in two halves. The m hypersurfaces divide the ball in 2^m or more open quadrants. These quadrants can be divided into two groups, namely, *Strictly Non-supporting Quadrants* and *Strictly Supporting Quadrants.*

Definition 3.7 *A point* s′ *is in a* Strictly Non-supporting Quadrant *of a ball* $B_r(\mathrm{s})$ *of a point* $\mathrm{s} \in \mathcal{T}^j_{i_1 \ldots i_m}$ *if there exists* $\epsilon_{i_k} < 0, k = 1, m$ *or* $\epsilon_{i_k} > 0, k = 1, m$ *such that*

$$\mathrm{s}' \in \alpha_{i_k}(X_{i_k})^{-1}(\epsilon_{i_k}), \forall k = 1, m.$$

The quadrant $B_r(\mathrm{s}) \cap \mathring{\mathcal{Z}}_j$ is a *strictly non-supporting quadrant.* If ϵ_{i_k}'s in the definition of a Strict non-supporting quadrant are allowed to be zero, we have a *Non-supporting Quadrant.* In particular, $B_r(\mathrm{s}) \cap \mathcal{Z}_j$ is a *non-supporting quadrant.* The complement of non-supporting quadrants in the ball are *Strictly Supporting Quadrants.* A corresponding definition of a *Supporting Quadrant* is as follows:

Definition 3.8 *A* Supporting Quadrant *of ball* B_r *of a point* $\mathrm{s} \in \mathcal{T}^j_{i_1 \ldots i_m}$ *consists of the* Strict Supporting Quadrant *and* $\cup_{k \in i_1 \ldots i_m} \mathcal{T}^j_k \setminus \{\cap_{k \in i_1 \ldots i_m} \mathcal{T}^j_k\}$.

A Supporting Quadrant of a point $\mathrm{s} \in \mathcal{T}^j_i$ is null, since its strict supporting quadrant is null and the set $\mathcal{T}^j_i \setminus \mathcal{T}^j_i$ is also null.

Proposition 3.10 *Let* s′ *be a point in the* Supporting Quadrant *of a point* $\mathrm{s} \in \mathcal{T}^j_{i_1 \ldots i_m}$. *Consider the tangent space* $T_{\mathrm{s}'}M$. *There exists a supporting hyperspace of* $F^0(\mathrm{s}')$ *that contains the face* $\mathcal{P}_{i_1 \ldots i_m}(\mathrm{s}')$ *of* $F^0(\mathrm{s}')$.

Proof: We show that if a hyperspace contains the segment $\mathcal{P}_{i_1 \ldots i_m}$, then $\alpha_k(X_k), k = i_1 \ldots i_m$, are not all strictly positive or negative. By definition, this is the condition for a point to be in a supporting quadrant and hence the claim.

Consider $\beta \in T^*_{\mathrm{s}'}M$. Consider a plane $\mathcal{P} = \{0, \beta\}$ through the origin that is supporting to the convex set defined by $\mathcal{P}_k, k = i_1 \ldots i_m$ and containing $\mathcal{P}_{i_1 \ldots i_m}(\mathrm{s}')$.

Since, β contains the kernel of all the planes $\alpha_k, k = i_1 \dots i_m$, and is supporting to the convex set, it follows that $\beta = \sum_{k=i_1 \dots i_m} c_k \alpha_k$, $c_k \geq 0$, not all c_k zero and for a point $\mathbf{v}$ in $F^0(\mathbf{s}')$, $\beta(\mathbf{v}) \geq 0$. If point $\mathbf{v}$ lies on $\mathcal{P}_{i_1 \dots i_m}(\mathbf{s}')$, then $\alpha_k(\mathbf{v}) = \alpha_k(X_k)$, and $\beta(\mathbf{v}) = 0$. So, $\beta(\mathbf{v}) = \sum_{k=i_1 \dots i_m} c_k \alpha_k(\mathbf{v}) = \sum_{k=i_1 \dots i_m} c_k \alpha_k(X_k) = 0$, but since $c_k \geq 0$, not all zero, it follows that not all of the $\alpha_k(X_k)$ are strictly positive or negative. $\diamond$

3.2.10 Forward Projection of Singular Invariant Set

Now, we consider computing the forward projection of a singular invariant set. This boundary of the forward projection is also the boundary of perturbations of the unstable manifolds. The perturbations of the *stable manifold* is the forward projection of the negative field $\dot{\mathbf{s}} \in -F^0(\mathbf{s})$.

Consider a point $\mathbf{s}$ on a face $\mathcal{T}^j_{i_1 \dots i_m}$ of codimension m on the boundary of a *singular invariant set* $\delta \mathcal{Z}_j$. Consider any *face preserving* parametrization of the boundary of the polytope F^0 such as parametrized vector field in equation 3.4 in a ball $B_r(\mathbf{s})$, of radius $r > 0$ centered at $\mathbf{s}$. Let X denote a constant parameter field with $X \in \mathcal{P}_{i_1 \dots i_m}$ in the ball $B_r(\mathbf{s})$ such that $X(\mathbf{s}) = 0$, and $\mathbf{s}$ is a hyperbolic singularity of X. Let α be defined as in equation 3.21 for any i_k, and i_l, $k, l \in 1 \dots m$. Since $X \in \mathcal{P}_{i_1 \dots i_m}$, $\alpha(X) \equiv 0$. Also, if $X \in \mathcal{P}_{i_1 \dots i_m}$, then a specification $\{X_{i_k}, \alpha_{i_k}\}$ of a hyperplane $\mathcal{P}_{i_k}$ is also equivalent to $\{X, \alpha_{i_k}\}$ for $\forall k \in 1, \dots, m$.

Let the dimension of Unstable Subspace $E^u(\mathbf{s})$ be p. If $A = DX(\mathbf{s})$ is the linearization of the field at the singularity, then the number of eigenvalues of A with positive and negative real parts are equal to p and $(n-p)$ respectively. The linear map A has a unique decomposition $T_{\mathbf{s}}M = E^u \oplus E^s$ so that A is expansion on the subspace E^u of dimension p and a contraction on E^s of dimension $(n-p)$. Denote $A_u = A|E^u$ and $A_s = A|E^s$ [HirSma 74]. Consider a further splitting of A_s(or similarly of A_u) as $A_s = \{A_s^1, \dots, A_s^r\}$, where each $A_s^i = A_s|_{E_i^s}$ of size $q_i \times q_i$, is an *elementary Jordan*

Matrix of the form

$$\begin{bmatrix} \lambda & & & & & \\ 1 & \lambda & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & 1 & \lambda & \\ & & & & 1 & \lambda \end{bmatrix},$$

on the diagonal of A_s, so that $E^s = E_1^s \oplus \ldots \oplus E_r^s$. Consider a matrix A_s^i corresponding to a real eigenvalue λ. It is easy to notice that if $\{e_i^k, k = 1, q_i\}$ is the basis of E_i^s that brings A_s^i into an elementary Jordan matrix form, then the subspaces defined by $\{e_i^k, k \geq k_0\}, 1 \leq k_0 \leq q_i$ are invariant subspaces. In particular, $\{e_i^k, k \geq 2\}$ is a $(q_i - 1)$-dimensional invariant subspace.

Consider now Proposition 1.1. The proof follows several lemmas. Eventually, we show that the hyperplane determined by $E^u \oplus E_{-1}^s$ is a supporting hyperplane of $F^0(\mathrm{s})$. In addition, we also prove that $\mathcal{Z}_j$ and $F^0(\mathrm{s})$ lie in the same half space of the hyperplane.

Now, consider combinatorial implication of the proposition on the dimensions of the stable, unstable subspaces, and the dimension of the set $\mathcal{T}_{i_1 \ldots i_m}^j$. If $dim(E^u) = p$, then $dim(E^s) = n - p$. The dimension of invariant subspace E_{-1}^s is $(n - p - 1)$. Since, subspace E_{-1}^s is a subset of $T_{\mathbf{s}} \mathcal{T}_{i_1 \ldots i_m}^j$, the dimension of the tangent space of the face $\mathcal{T}_{i_1 \ldots i_m}^j$ on which s lies is equal to or greater than $(n - p - 1)$. There can be at most $(p + 1)$ hypersurfaces meeting at the point, *i.e.*, $m \leq p + 1$. The one-dimensional subspace $E^s \ominus E_{-1}^s$ which does not belong to the tangent space $T\mathcal{T}_{i_1 \ldots i_m}^j$ must correspond to a real eigenvalue. It can not belong to a complex eigenvalue for their invariant subspaces are of even dimensions and, therefore, E_{-1}^s can not be an invariant subspace. As an example, consider a case when $E^s \ominus E_{-1}^s$ corresponds to a one-dimensional subspace of a real eigenvalue which has just one elementary Jordan block. Then, it must be the basis e_i^1 of the subspace E_i^s of the elementary Jordan matrix because, then only the remaining $(n - p - 1)$-dimensional subspace of the stable subspace remains invariant.

Lemma 3.2 *Let q-dimensional invariant subspace be contained in the tangent set* $T\mathcal{T}^j_{i_1\ldots i_m}$, *where* $m \leq n-q$, *then the face* $\mathcal{P}_{i_1\ldots i_m} \subset T\mathcal{T}^j_{i_1\ldots i_m}$.

Proof: The tangent space $T\mathcal{T}^j_{i_1\ldots i_m}$ is defined as $\cap_{k=i_1\ldots i_m} ker(D\mathcal{P}_k)$. The lemma asserts that the subspace $\cap_{k=i_1\ldots i_m}\mathcal{P}_k$ is a subset of $\cap_{k=i_1\ldots i_m} ker(D\mathcal{P}_k)$. The derivative $D\mathcal{P}_k$ of transition hypersurfaces and the hyperspaces $\mathcal{P}_k = \{X_k, \alpha_k\}$ are related by equation 3.22, which reduces to $[\alpha_k]\,[DX] = D\mathcal{P}_k$ at a singular point. If q-dimensional subspace W^q is a subset of $T\mathcal{T}^j_{i_1\ldots i_m}$, then $D\mathcal{P}_k(W^q) = 0, k = i_1 \ldots i_m$. Since W^q is also invariant subspace of DX, it follows that $\alpha_k(DX(W^q)) = \alpha_k(W^q) = 0, k = i_1 \ldots i_m$.
◇

The following lemma shows that the subspaces $E^u \oplus E^s_{-1}$ and E^s of $T_{\mathbf{s}}M$, for $\mathbf{s} \in \mathcal{T}^j_{i_1\ldots i_m}$ and $X(\mathbf{s}) = 0$, and $\mathbf{s}$ a hyperbolic singularity, are determined independent of how X is chosen on the face $\mathcal{P}_{i_1\ldots i_m}$. Recall that the subspace E^s_{-1} is determined to lie in the tangent space, $T\mathcal{T}^j_{i_1\ldots i_m}$. However, the eigenspaces of E^s_{-1} and the eigenspace of $E^s \ominus E^s_{-1}$ may rearrange in the subspace E^s due to different choices of X. Nevertheless, some subspaces are determined independent of such a choice (face preserving parameterization as in equation 3.4).

Lemma 3.3 *The subspaces* $E^u \oplus E^s_{-1}$ *and* E^s *of* $T_{\mathbf{s}}M$ *are determined independent of any parametrization if the conditions of Proposition 1.1 are satisfied.*

Proof: Consider a chart (U, x) of M, $\mathbf{s} \in U$, $x(\mathbf{s}) = 0$, which has *submanifold property, i.e.,*

$$x: U \to \mathbf{E} \times \mathbf{F}; \quad x(U \cap \mathcal{T}^j_{i_1\ldots i_m}) = \mathbf{E} \times \{0\}.$$

The dimensions of the real vector spaces $\mathbf{E}$ and $\mathbf{F}$ are $(n-m)$ and m respectively. Let the matrix of the linear map $A = DX(\mathbf{s})$ in the chart be denoted $A' = \begin{bmatrix} J_{n-m\times n-m} & K_{n-m\times m} \\ L_{m\times n-m} & M_{m\times m} \end{bmatrix}$, where X is a cone boundary field lying in $\mathcal{P}_{i_1\ldots i_m}$ with any parametrization so that $X(\mathbf{s}) = 0$, and $\mathbf{s}$ is a hyperbolic singularity. For notational convenience, denote the face $\mathcal{P}_{i_1\ldots i_m}$ by $\mathcal{P}_{1\ldots m}$ by renumbering, if necessary. Next, we show the structure of the matrix A'.

Choose coordinate maps $x^{n-m+i} = \alpha_i(X_i), i = 1, m$ as a basis of the vector space $\mathbf{F}$. This is possible, since submanifold $\mathcal{T}^j_{1\ldots m}$ is given as $\cap_{i=1,m}(x^{n-m+i})^{-1}(0) =$

$\cap_{i=1,m}(\alpha_{i_k}(X_{i_k}))^{-1}(0)$. Let the induced basis on TM be denoted β. The matrix of $D\mathcal{P}_i = D(\alpha_i(X_i)) = Dx^{n-m+i}, i = 1, m$, in this basis is given as $D_\beta = [0 \quad I]$, where 0 is $m \times (n-m)$ zero matrix and I is $m \times m$ identity matrix. Let the matrix of hyperspaces $\mathcal{P}_i = \{0, \alpha_i\}, i = 1, m$, in this basis be $C_\beta = [\alpha_{ij}]$. By Lemma 3.2, the kernel of $\alpha_i, i = 1, m$, and $D\mathcal{P}_i, i = 1, m$, are the same, so $\alpha_{ij} = 0, j = 1, n-m, i = 1, m$. So, $C_\beta = [0 \quad N]$, where 0 is $m \times (n-m)$ zero matrix and $N = [\alpha_{ij}], j = n-m+1, n, i = 1, m$. From $C_\beta A' = D_\beta$ of equation 3.22, it follows that $NL = 0$ and $NM = I$. Since, the α's are non-vanishing, N is non-zero. Therefore, $L = 0$ and $M = N^{-1}$. Notice that subblock M is determined independent of any parametrization of X.

Now, let $\mathbf{v}$ be elements of $T\mathbf{E}$ and $\mathbf{w}$ those of $T\mathbf{F}$. The structure so determined, of A', makes the tangent space of the submanifold $\mathcal{T}^j_{1\ldots m}$ the invariant subspace E^s_{-1}, as it should since an element $(\mathbf{v}, 0)$ in $T_0\mathbf{E}$ is mapped to $(J\mathbf{v}, 0)$ also belonging to $T_0\mathbf{E}$. Now, for a vector $(\mathbf{v}, \mathbf{w}), \mathbf{w} \neq 0$ to be an eigenvector of A', $\mathbf{w}$ must be an eigenvector of M. But M is determined independent of any parametrization of X and, therefore, subspaces $E^u \oplus E^s_{-1}$ and E^s of $T_{\mathbf{s}}M$ are determined. $\diamond$

Let $\beta = \{e_i, i = 1, n\}$ be a basis of the tangent space $T_{\mathbf{s}}M$. Let $<,>_\beta$ be a inner product that makes the basis β of $T_{\mathbf{s}}M$ an orthonormal basis and $<,>^*$ be the induced inner product on $T^*_{\mathbf{s}}M$ using the dual basis. Consider a representation in dual coordinates $\{e^*_i, i = 1, n\}$ of elements $\alpha_i = \alpha_{i1}e^*_1 + \alpha_{i2}e^*_2 + \ldots + \alpha_{in}e^*_n \in T^*_{\mathbf{s}}M$. Let the corresponding matrix B be $[\alpha_{ij}]$. The following lemma for some elements of the inverse of matrix B is based on a property of convex sets.

Lemma 3.4 *Let the matrix $B^{-1} = [\eta_{ij}]$ in the basis β be the inverse of a full rank matrix $B = [\alpha_{ij}]$. Let for some j_0, $\alpha_{ij_0} \geq 0, i = 1, n$, and $\{0, \alpha_i\}, i = 1, n$, bound a convex set, then $\eta_{j_0 i} \geq 0, i = 1, n$.*

Proof: Let $(n-1)$ independent elements $\mathbf{v}_i, i = 1, n-1$, span a hyperplane W of $\mathbf{R}^n$. Define a one form $\gamma \in \mathbf{R}^{n^*}$ by $\gamma(\mathbf{v}) = det[\mathbf{v}_1 \ldots \mathbf{v}_{n-1} \ \mathbf{v}]^T$. There exists a unique $\mathbf{w} \in \mathbf{R}^n$ such that $<\mathbf{v}, \mathbf{w}> = \gamma(\mathbf{v})$. Vector $\mathbf{w}$ is *cross-product* of elements $\mathbf{v}_i, i = 1, n-1$, - a generalization of the usual *cross-product* in three space.

Without loss of generality, assume that $\alpha_{ij_o} = <\alpha_i, e^*_{j_0}>^*, i = 1, n-1$, are not all zero. Otherwise, a renumbering of $\alpha_i, i = 1, n$, will work because they are linearly

independent so not all α_{ij_o} are zero. Consider an $(n-1)$-dimensional hyperplane $W \subset T^*_s M$ spanned by linearly independent set $\alpha'_i = \alpha_i - <\alpha_i, e^*_{j0}>^* e^*_{j0}, i = 1, n-1$. Define a corresponding one form γ. It follows that $\gamma(\mathbf{v}) = c <\mathbf{v}, e^*_{j0}>^*$, $c \in \mathbf{R}$. The elements of B^{-1}, the inverse of B, are $\eta_{j_0 k} = \frac{1}{det(B)} B_{kj_0}$, where B_{kj_0} are co-factor matrices of matrix B. The co-factor B_{kj_0} is

$$det \begin{bmatrix} \alpha'_1 \\ \cdot \\ \underbrace{e^*_{j0}}_{k^{th}\, row} \\ \cdot \\ \alpha'_n \end{bmatrix}.$$

With the assumption that $\{0, \alpha_i\}, i = 1, n$, bound a convex set, we show later that $B_{kj_0} = b_k \gamma(e^*_{j0}), b_k \geq 0$. Now, $\gamma(e^*_{j0}) = c <e^*_{j0}, e^*_{j0}>^* = c$. Hence, $\eta_{j_0 k} = \frac{1}{det(B)} B_{kj_0} = \frac{b_k c}{det(B)}$. Since, diagonal terms of $B^{-1}B$ are equal to one, consider the $j_0 j_0^{th}$ term $1 = \sum_k \eta_{j_0 k} \alpha_{kj_0} = \frac{c}{det(B)} \sum_k b_k \alpha_{kj_0}$. Since, all $\alpha_{kj_0} \geq 0$, and $b_k \geq 0$, it follows that $\frac{c}{det(B)}$ must also be positive. Hence, $\eta_{j_0 k} = \frac{c}{det(B)} b_k$ are all greater than or equal to zero. Now to see that $B_{kj_0} = b_k \gamma(e^*_{j0}), b_k \geq 0$, consider the following two cases.

1. If $k = n$, $B_{kj_0} = \gamma(e^*_{j0})$, so $b_n = 1$.

2. For $k < n$: Take e^*_{j0} in the n^{th} row of $\gamma(e^*_{j0})$ to the k^{th} row which is equivalent to $(-1)^{n-k}\gamma(e^*_{j0})$. This leaves $\alpha'_k \ldots \alpha'_{n-1}$ as the last $(n-k-1)$ rows. Taking α_k in the $(k+1)^{th}$ row to the n^{th} row is equivalent to setting $(-1)^{n-k-1}(-1)^{n-k}\gamma(e^*_{j0}) = (-1)^{2(n-k)-1}\gamma(e^*_{j0})$. Now, to replace α'_k in the n^{th} row by α'_n, consider the following. The $(n-1)$ plane spanned by $ker(e^*_{j0})$ is supporting hyperplane to the convex set $\{0, \alpha_i\}, i = 1, n$, because from Proposition 3.9, $< \alpha_i, e^*_{j0} >^* = \alpha_{ij_0} \geq 0, i = 1, n$. There exist constants $a_i, i = 1, n$, so that $\sum_{i=1,n} a_i \alpha_i = e^*_{j0} \Rightarrow \sum_{i=1,n} a_i \alpha'_i + \sum_{i=1,n} a_i < \alpha_i, e^*_{j0} >^* e^*_{j0} = e^*_{j0} \Rightarrow \sum_{i=1,n} a_i \alpha'_i = (1 - \sum_{i=1,n} a_i < \alpha_i, e^*_{j0} >^*) e^*_{j0} \Rightarrow \sum_{i=1,n} a_i \alpha'_i = 0 \Rightarrow \alpha'_n = -\sum_{i=1,n-1} \frac{a_i}{a_n} \alpha'_i$. Constant a_n is not equal to zero, because otherwise the relation in the fourth statement would make $\alpha_i, i = 1, n-1$, linearly dependent. The implication from the third statement about the fourth follows from an expression obtained by taking the inner product of the first statement with e^*_{j0}. Replacing α'_k in the n^{th} row

by α'_n makes it equal to B_{kj_0}. So, in total $B_{kj_0} = (-\frac{a_k}{a_n})(-1)^{2(n-k)-1}\gamma(e^*_{j_0}) = (\frac{a_k}{a_n})(-1)^{2(n-k-1)}\gamma(e^*_{j_0}) = (\frac{a_k}{a_n})\gamma(e^*_{j_0})$, so $b_k = \frac{a_k}{a_n} \geq 0$.

$\diamond$

Lemma 3.5 *Let* $\mathbf{s} \in \mathcal{T}^j_{k_1 \dots k_m}$ *where the conditions of Proposition 1.1 are satisfied. Then, the* C^0 *neighborhood* F^0 *given by* $\{\mathcal{P}_i = \{0, \alpha_i\}, i \in \{k_1 \dots k_m\}\}$ *at this point and the convex set* $\{\{0, -D\mathcal{P}_i\}, i \in \{k_1 \dots k_m\}\}$ *lie in the same half space of the supporting hyperspace* $E^u \oplus E^s_{-1}$.

Proof: The proof is divided into two parts. First, we consider the case when $m = n$, so that $\mathbf{s}$ is a vertex on $\delta\mathcal{Z}_j$. Then, we prove the general case. Consider the splitting of the linear map $A = DX$, so that $A_s = A|E^s$ and $A_u = A|E^u$ are the linear maps on the stable and unstable subspaces, and $T_\mathbf{s}M = E^s \oplus E^u$. Consider a basis $\beta = \{e^s, e_j, j = 1, n-1\}$, of $T_\mathbf{s}M$ induced by the eigenspaces of the linear map A, where e^s corresponds to the one-dimensional subspace of a real eigenvalue not contained in the tangent space $T\mathcal{T}^j_{k_1 \dots k_m}$, as in the conditions of Proposition 1.1. Let $\{e^{s^*}, e^*_j, j = 1, n-1\}$ be the corresponding dual basis of $T^*_\mathbf{s}M$. Consider an inner product $<,>_\beta$, and $<,>^*_\beta$ that make these basis into an orthonormal basis.

If $\mathbf{s}$ is a vertex, then $dim(E^u) = (n-1)$, $dim(E^s) = 1$, and $A_s = -\lambda, \lambda > 0$. The codimension one subspace E^s_{-1} of the stable subspace is the trivial element 0. Subspace $ker(e^{s^*}) = E^u \oplus E^s_{-1}$ is supporting to $\mathcal{Z}_j$. Therefore, by Definition 3.5 and Proposition 3.9, it follows that an orientation of e^{s^*} can be chosen so that $<e^{s^*}, D\mathcal{P}_i>^*_\beta \geq 0, i \in \{k_1 \dots k_n\}$. Let the matrix of hyperspaces $\mathcal{P}_i = \{0, \alpha_i\}, i \in \{k_1 \dots k_n\}$, in this basis be $C_\beta = [\alpha_{ij}]$ and the matrix of $D\mathcal{P}_i, i \in \{k_1 \dots k_n\}$, be $D_\beta = [\eta_{ij}]$. If the basis β expression of the linear map $A = DX$ is denoted A_β, then relations $C_\beta A_\beta = D_\beta \Rightarrow C^{-1}_\beta = A_\beta D^{-1}_\beta$ follow from equation 3.22.

By the sign convention in equation 3.19, all elements $\mathbf{v} \in T_\mathbf{s}M$, so that $\alpha_i(\mathbf{v}) \geq 0, i \in \{k_1 \dots k_n\}$ belong to $\{\mathcal{P}_i, i \in \{k_1 \dots k_n\}\}$. In basis β, this implies that there exists an element $\mathbf{z}_\beta = [z_1 \dots z_n]^T, z_i \geq 0$, so that $C_\beta \mathbf{v}_\beta = \mathbf{z}_\beta$ for $\mathbf{v}_\beta \in \{\mathcal{P}_i, i \in \{k_1 \dots k_n\}\}$. Then we show later that such elements $\mathbf{v}$ also satisfy the condition $e^{s^*}(\mathbf{v}) \leq 0$. Therefore, by equation 3.20, $\{0, -e^{s^*}\}$ is a supporting hyperplane of $\{\mathcal{P}_i, i \in \{k_1 \dots k_n\}\}$. Now, by construction, $<-e^{s^*}, -D\mathcal{P}_i>^*_\beta = <e^{s^*}, D\mathcal{P}_i>^*_\beta \geq 0, i \in$

$\{k_1 \ldots k_n\}$. Hence, by Proposition 3.9, $\{0, -e^{s^*}\}$ is also the supporting hyperspace to $\{\{0, -D\mathcal{P}_k\}, i \in \{k_1 \ldots k_n\}\}$. Therefore, these two sets lie in the same half space.

Now, to see that $e^{s^*}(\mathbf{v}) \leq 0$, we use the basis β expressions.

$$\begin{aligned} e^{s^*}(\mathbf{v}) &\equiv e^{s^*}(C_\beta^{-1}\mathbf{z}_\beta) \\ &\equiv e^{s^*}(A_\beta D_\beta^{-1}\mathbf{z}_\beta) \\ &\equiv e^{s^*}\left(\begin{bmatrix} -\lambda & 0 \\ 0 & A_u \end{bmatrix} D_\beta^{-1}\mathbf{z}_\beta\right) \end{aligned}$$

Since, $<e^{s^*}, D\mathcal{P}_i>^*_\beta \geq 0, i \in \{k_1 \ldots k_n\}$, the first row of D_β^{-1} is composed of non-negative elements by Lemma 3.4. It follows that the first row of $D_\beta^{-1}\mathbf{z}_\beta = a$, a a constant, is positive. Therefore, the first row of $A_\beta D_\beta^{-1}\mathbf{z}_\beta = -\lambda a$. So, $e^{s^*}(\mathbf{v}) = [1\,0 \ldots 0][-\lambda a \ldots]^T = -\lambda a \leq 0$.

Now, consider the case when $m < n$. Introduce fictitious $D\mathcal{P}_i, i \in \{k_{m+1} \ldots k_n\}$ so that $<e^{s^*}, D\mathcal{P}_i>^*_\beta = 0, i \in \{k_{m+1} \ldots k_n\}$, and together with $\{D\mathcal{P}_i, i = k_1 \ldots k_m\}$ they form an n-dimensional linearly independent set. To see that such a choice is possible, we show by contradiction a claim that at least one of the existing hyperplanes have a non-zero component along e^{s^*}, *i.e.*, $<e^{s^*}, D\mathcal{P}_i>^*_\beta \neq 0, i \in \{k_1 \ldots k_m\}$. So, other planes can be chosen with zero e^{s^*} component. By assumption, $ker(e^{s^*}) = E^u \oplus E^s_{-1} \supset TT^j_{k_1 \ldots k_m} = \cap_{i \in k_1 \ldots k_m} ker(D\mathcal{P}_i)$. If $<e^{s^*}, D\mathcal{P}_i>^*_\beta = 0, i \in \{k_1 \ldots k_m\}$, then $e^s \in \cap_{i \in k_1 \ldots k_m} D\mathcal{P}_i$. But then, e^s is not in the kernel of e^{s^*} because $e^{s^*}(e^s) = 1$.

First, consider the case when only one fictitious hyperplane $D\mathcal{P}_{m+1}$ is introduced. The convex set $\{\{0, D\mathcal{P}_i\}, i = k_1 \ldots k_m\}$ is the union of the convex sets $\{\{0, D\mathcal{P}_{m+1}\}, \{0, D\mathcal{P}_i\}, i = k_1 \ldots k_m\} \cup \{\{0, -D\mathcal{P}_{m+1}\}, \{0, D\mathcal{P}_i\}, i = k_1 \ldots k_m\}$. As an extension, if there are $(n - m)$ such fictitious hyperplanes, then the convex set $\{\{0, D\mathcal{P}_i\}, i = k_1 \ldots k_m\}$ is the union of all convex sets which contain both negative and positive half spaces of all the fictitious planes defined as in the case of one fictitious plane. If all such convex components lie in the same half space of $E^u \oplus E^s_{-1}$, then it follows that their union $\{\{0, D\mathcal{P}_i\}, i = k_1 \ldots k_m\}$ also does. Now, consider one such convex set. Let the matrix of $D\mathcal{P}_i, i \in \{k_1 \ldots k_n\}$, be $D = [\eta_{ij}]$. The fictitious planes $D\mathcal{P}_i, i \in \{k_{m+1} \ldots k_n\}$, appear either as themselves or negated. In either case, their component along e^{s^*} is zero *i.e.*, $<e^{s^*}, D\mathcal{P}_i>^*_\beta = 0, i \in \{k_{m+1} \ldots k_n\}$,

so that $D = [\eta_{ij}]$ satisfies the conditions of Lemma 3.4. Now, if we add fictitious $\alpha_i = D\mathcal{P}_i A^{-1}, i = k_{m+1} \dots k_n$, then by the same argument, the convex set $\{\{0, \alpha_i\}, i = k_1 \dots k_m\}$ is the union of all convex sets which contain both negative and positive half spaces of all the fictitious planes $\{0, \alpha_i\}, i = k_{m+1} \dots k_n$. But, we already proved that each such convex set lies in the same half space of $E^u \oplus E^s_{-1}$. ◇

Proof of Proposition 1.1: The subspace E^u is the supporting subspace to $\mathcal{Z}_j$ and, therefore, $E^u \subset \mathcal{P}$ for $\mathcal{P} \in \mathcal{H}(\mathcal{T}^j_{i_1 \dots i_m}, \mathbf{s})$ (see Definition 3.5). The codimension one subset E^s_{-1} of the stable subspace is a subset of $T\mathcal{T}^j_{i_1 \dots i_m}$ and, therefore, is also contained in $\mathcal{P}$. The linear map $A = DX$ is hyperbolic and, hence, E^u and E^s_{-1} are linearly independent. Hence, a span of $\mathcal{P}$ is $E^u \oplus E^s_{-1}$. It follows from Lemma 3.5 that this hyperspace $\mathcal{P}$ is a supporting hyperspace of the set $\{\mathcal{P}_k, k = i_1, \dots i_m\} \subset T_{\mathbf{s}}M$. By definition of a supporting space to a convex set from equation 3.20, the supremum is zero and hence function H vanishes. Therefore, point $\mathbf{s}$ is a subset of the characteristic set $\mathcal{C}$ of function H. ◇

Dimensional consideration for Proposition 1.1 shows that it is valid for points on segments whose codimension $m \leq p + 1$. A corollary below shows that a similar assertion follows for points on a segment whose codimension $m > p + 1$.

Proof of Corollary 1.1: Consider a point $\mathbf{s}$ in the intersection $\mathcal{T}^j_{i_1 \dots i_m} \cap \mathcal{T}^j_{k_1 \dots k_q}$, and a sequence of points $\{\mathbf{s}_i\}, i = 0, \infty$, in $\mathcal{T}^j_{i_1 \dots i_m}$ and another sequence $\{\mathbf{s}'_k\}, k = 0, \infty$, in $\mathcal{T}^j_{k_1 \dots k_q}$. Let both sequences converge to point $\mathbf{s}$ in the intersection. Consider the supporting hyperplanes $E^s_{-1}(\mathbf{s}_i) \oplus E^u(\mathbf{s}_i)$ and $E^s_{-1}(\mathbf{s}'_k) \oplus E^u(\mathbf{s}'_k)$ for points $\mathbf{s}_i$ in $\mathcal{T}^j_{i_1 \dots i_m}$ and $\mathbf{s}'_k$ in $\mathcal{T}^j_{k_1 \dots k_q}$, respectively. Since field Y changes continuously on all segments of $\mathcal{Z}_j$, the unstable subspace and stable subspace change continuously. Hence, $E^u(\mathbf{s}_i)$ and $E^u(\mathbf{s}'_k)$ converge to the same subspace $E^u(\mathbf{s})$. The corresponding stable subspaces also converge to the same subspace. However, the codimension one subspace $E^s_{-1}(\mathbf{s}_i)$ and $E^s_{-1}(\mathbf{s}'_k)$ which are subset of the tangent spaces $T\mathcal{T}^j_{i_1 \dots i_m}$ and $T\mathcal{T}^j_{k_1 \dots k_q}$, converge to different subspaces. Let Hamiltonian H vanish for $\beta_i \in T^*M$ at $\mathbf{s}_i$ where $E^s_{-1}(\mathbf{s}_i) \oplus E^u(\mathbf{s}_i) = ker(\beta_i)$. It also vanishes for $\gamma_k \in T^*M$ at $\mathbf{s}'_k$ where $E^s_{-1}(\mathbf{s}'_k) \oplus E^u(\mathbf{s}'_k) = ker(\gamma_k)$. Let $\beta_i \stackrel{i \to \infty}{\Longrightarrow} \beta$ and $\gamma_k \stackrel{k \to \infty}{\Longrightarrow} \gamma$. Then, H vanishes at $\mathbf{s}$ for both β and γ.◇

Proof of Theorem 1.2: The proof is in several steps. First notice that since Λ_j is a subset of the boundary of the singular invariant set $\delta\mathcal{Z}_j$, it is a union of

proper faces. Therefore, at least $p+1$ transition hypersurfaces define the faces of Λ_j, *i.e.*, codimension $m \geq p+1$. To see this, note that the unstable subspace E^u is a strictly supporting subspace, so by Definition 3.6 $E^u \cap T_{\mathbf{s}}\mathcal{T}^j_{i_1 \ldots i_m} = 0$ at a point $\mathbf{s} \in \mathcal{T}^j_{i_1 \ldots i_m}$. Since, $q = dim(\mathcal{T}^j_{i_1 \ldots i_m}) \leq n-1$ and $p = dim(E^u) \leq n-1$, we find $p+q \leq n-1 \Rightarrow q \leq n-p-1$. Hence, $m = n-q \geq p+1$.

Since, the conditions of Proposition 1.1, or Corollary 1.1 are satisfied at all points of Λ_j, it follows that it is a subset of the characteristic set $\mathcal{C}$. Hence, Λ_j is an *initial manifold.*

The local unstable manifold $W^u_r(\mathbf{s})$ is an image of E^u by a continuous map from the tangent space $T_{\mathbf{s}}M$ to M [PalMel 82, pp. 73]. And since the intersection $E^u \cap T_{\mathbf{s}}\mathcal{T}^j_{i_1 \ldots i_m}$ of the unstable subspace and the tangent space, is the trivial element, there exists a $r > 0$ so that in a ball $B_r(\mathbf{s})$, the intersection of $W^u_r(\mathbf{s})$ and the transition hypersurfaces $\mathcal{T}^j_k, k = i_1 \ldots i_m$, is isolated. In particular, $W^u_r(\mathbf{s})$ is a subset of the supporting quadrants of ball $B_r(\mathbf{s})$ where characteristic flow of the cone boundary surface is well defined by Proposition 3.10 and Proposition 3.6. The characteristic directions of trajectories in the unstable manifold span E^u as they approach $\mathbf{s}$ for negative time. And since the tangent space to the unstable manifold $W^u_r(\mathbf{s})$ at $\mathbf{s}$ is E^u, which has trivial intersection with the tangent space $T_{\mathbf{s}}\mathcal{T}^j_{i_1 \ldots i_m}$, Theorem 3.6 implies that the integral manifold exists and is uniquely defined. The p-dimensional unstable manifold and the $(n-p-1)$-dimensional Λ_j together span the $(n-1)$-dimensional surface. Inside ball $B_r(\mathbf{s})$, the integral manifold is, therefore, the boundary of the forward projection.$\diamond$

This proposition describes a generic situation in that Λ_j consists of faces of codimension greater than or equal to $p+1$, and E^u is considered strictly supporting. Example 3.4 is such an illustration. However, if there are faces of codimension smaller than $p+1$ in Λ_j or E^u is not strictly supporting, then the local unstable manifold W^u_r of points on such faces must make the face invariant for the forward projection to be well defined, as perhaps should be obvious from the proof above. Example 3.5 shows the boundary of the forward projection of the singular invariant set for such a degenerate case.

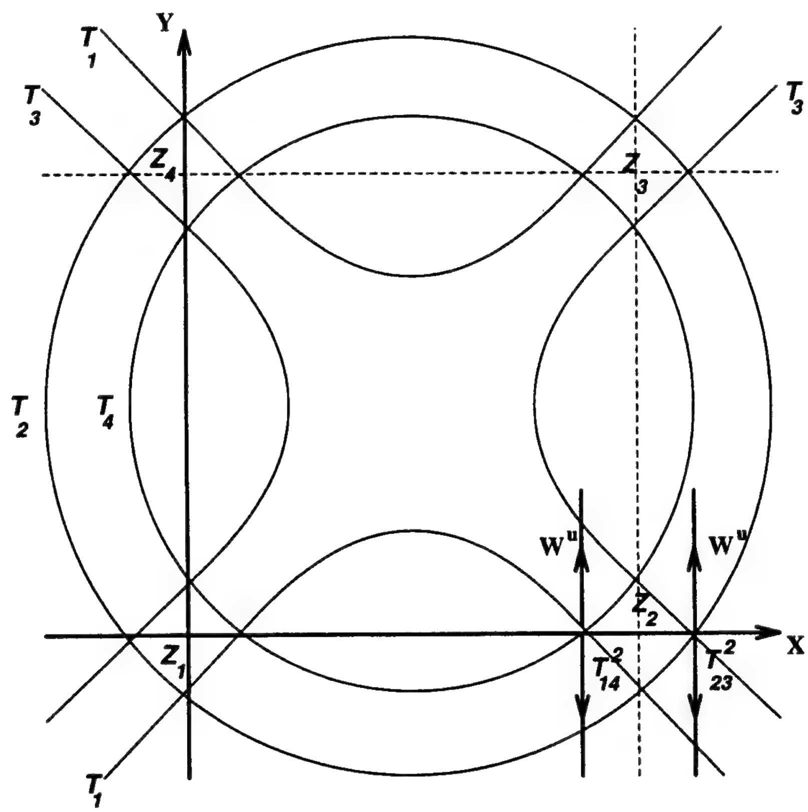


Figure 3.3: Planar Field in Example A

3.2.11 Examples of the Boundary of Forward Projection of Singular Invariant Set

Example 3.4 (A) Consider example 1.1. The *transition hypersurfaces* $\mathcal{T}_i$ corresponding to a hyperplane $\mathcal{P}_i = \{X_i, \alpha_i\}$ that bounds the convex polytope F^0 is defined by the zero set of a function $\alpha_i(X_i)$. They are $\mathcal{T}_1 \equiv -\epsilon_y(x^2-x)+\epsilon_x(y^2-y)-\epsilon_x\epsilon_y = 0$, $\mathcal{T}_2 \equiv \epsilon_y(x^2-x)+\epsilon_x(y^2-y)-\epsilon_x\epsilon_y = 0$, $\mathcal{T}_3 \equiv \epsilon_y(x^2-x)-\epsilon_x(y^2-y)-\epsilon_x\epsilon_y = 0$, and $\mathcal{T}_4 \equiv -\epsilon_y(x^2-x)-\epsilon_x(y^2-y)-\epsilon_x\epsilon_y = 0$. The transition hypersurfaces $\mathcal{T}_1$ and $\mathcal{T}_3$ are hyperbolas and $\mathcal{T}_2$ and $\mathcal{T}_4$ are ellipses. Consider the forward projection of the saddle singular invariant set $\mathcal{Z}_2$. Consider the vertex $\mathbf{s} = \mathcal{T}_{14}^2$ on the boundary $\delta\mathcal{Z}_2$ as a candidate for a point where conditions of the Proposition 1.1 are satisfied. In this planar example, there is a unique field X such that $X \in \mathcal{P}_{14}$ in some local neighborhood of $\mathbf{s}$. This field is $X = (-(x^2-x)-\epsilon_x)\frac{\partial}{\partial x} - (y^2-y)\frac{\partial}{\partial y}$. It is possible to check that

$X(\mathrm{s}) = 0$. The linearization of the field at s is hyperbolic. The stable subspace, E^s, is along the x-axis and the unstable subspace, E^u, is along the y-axis. The codimension one subspace E^s_{-1} of the stable subspace is the trivial element which is contained in the tangent space of the vertex s, the unstable subspace E^u is a strictly supporting subspace and the remaining one-dimensional subspace, which is E^s itself, is not contained in the tangent space of the point. Hence, all conditions of the Proposition 1.1 are satisfied. It follows from Theorem 1.2 that the local unstable manifold, $W^u_r(\mathrm{s})$, for some $r > 0$ defines the boundary of the closed set $\mathcal{Z}_2$. In this example, it happens that the unstable manifolds, $W^u(\mathrm{s})$, is the global boundary of the forward projection of the singular invariant set $\mathcal{Z}_2$. The situation at the vertex $\mathcal{T}^2_{23}$ is symmetric.

The perturbations of the stable manifolds of $\mathcal{Z}_2$ is constructed analogously by considering the forward projection of the negative of this differential inclusion problem, *i.e.*, the unstable manifolds of the negative fields at the vertices $\mathcal{T}^2_{13}$ and $\mathcal{T}^2_{24}$.$\diamond$

Example 3.5 (B) Though this is an example when $M = \mathbf{R}^3$, its obvious extensions give examples in all Euclidean spaces of dimensions two or more. Consider a nominal vector field Y as in Example A. Let the neighborhood F^0 be a cube whose six faces are parallel to the axes:

$$\begin{aligned}
\mathcal{P}_1 &= \{(-(x^2-x)-\epsilon_x)\frac{\partial}{\partial x} - (y^2-y)\frac{\partial}{\partial y} - (z^2-z)\frac{\partial}{\partial z}, dx\}, \\
\mathcal{P}_2 &= \{-(x^2-x)\frac{\partial}{\partial x} + (-(y^2-y)+\epsilon_y)\frac{\partial}{\partial y} - (z^2-z)\frac{\partial}{\partial z}, -dy\}, \\
\mathcal{P}_3 &= \{(-(x^2-x)+\epsilon_x)\frac{\partial}{\partial x} - (y^2-y)\frac{\partial}{\partial y} - (z^2-z)\frac{\partial}{\partial z}, -dx\}, \\
\mathcal{P}_4 &= \{-(x^2-x)\frac{\partial}{\partial x} + (-(y^2-y)-\epsilon_y)\frac{\partial}{\partial y} - (z^2-z)\frac{\partial}{\partial z}, dy\}, \\
\mathcal{P}_5 &= \{-(x^2-x)\frac{\partial}{\partial x} - (y^2-y)\frac{\partial}{\partial y} + (-(z^2-z)-\epsilon_z)\frac{\partial}{\partial z}, dz\}, \\
\mathcal{P}_6 &= \{-(x^2-x)\frac{\partial}{\partial x} - (y^2-y)\frac{\partial}{\partial y} + (-(z^2-z)+\epsilon_z)\frac{\partial}{\partial z}, -dz\},
\end{aligned} \tag{3.25}$$

where $\epsilon_x, \epsilon_y, \epsilon_z > 0$ are constants. Each *transition hypersurface* is a two sheeted hyperplane defined by: $\mathcal{T}_1 \equiv -x^2+x-\epsilon_x = 0$, $\mathcal{T}_2 \equiv y^2-y-\epsilon_y = 0$, $\mathcal{T}_3 \equiv x^2-x-\epsilon_x = 0$,

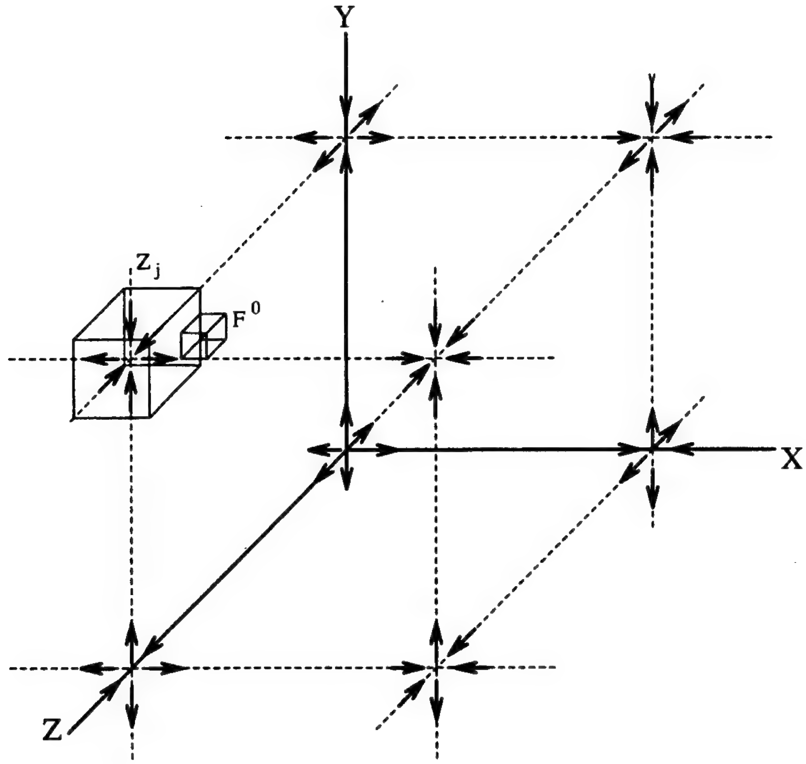


Figure 3.4: Example B

$\mathcal{T}_4 \equiv -y^2 + y - \epsilon_y = 0$, $\mathcal{T}_5 \equiv -z^2 + z - \epsilon_z = 0$, and $\mathcal{T}_6 \equiv z^2 - z - \epsilon_z = 0$. The *Singular Invariant Sets* $\mathcal{Z}_i, i = 1,8$ are centered at the eight vertices of the cube with body diagonal $(0,0,0)$ and $(1,1,1)$. The invariant subset around $(0,0,0)$ is a *source*, around $(1,1,1)$ is a *sink* and the remaining invariant subsets are *saddles.*

Consider the forward projection of the saddle at $(0,1,1)$. The unstable subspace E^u is one-dimensional along the x-axis. The stable subspace E^s is two-dimensional and is spanned by the y-axis and z-axis. The subset Λ_j of $\delta\mathcal{Z}_j$ where the conditions of Proposition 1.1 are satisfied consists of the transition hypersurfaces $\mathcal{T}_k^j, k = 2,4,5,6$ and all other segments in their closure. The local forward projection of $\mathcal{Z}_j$ is defined by the unstable manifold of these points. They are subsets of the transition hypersurfaces $\mathcal{T}_k^j, k = 2,4,5,6.\diamond$

Chapter 4

Conclusion

Fine Motion Planning is essential where the success of a goal-oriented behavior of a physical system is determined largely by uncertainties: in control equations that determine its evolution, in sensing that is used to determine the attainment of a goal, and in the shape or model of the environment that can only be engineered upto a prescribed accuracy. The domain of application of fine motion planning is assembly and dextrous manipulation of parts using mechanical manipulators, navigation of a mobile robot, goal-oriented behavior of a hopping robot, design of mechanical parts for ease in assembly, and design of mechanical parts feeder. In this thesis, we considered uncertainty in control and sensing.

4.1 Characterization of Control Uncertainty

In classical control theory, when a controller for a linear or a non-linear system is shown to be robust, the system rejects some disturbance or noise, stays stable and close to a nominal behavior. However in fine motion applications, the success of attaining the goal depends substantially on the perturbations from a nominal behavior. A few of the factors contributing to perturbations are errors in placement of parts relative to each other, tolerances in the shape and the size of the parts, the mechanics of motion in the presence of friction, transition from one contact type to another, deterministic but imprecisely known inertial properties of the system, and

non-deterministic sensor measurement noise.

If a nominal behavior is described by assigning a velocity (acceleration) to each state, then all possible perturbations can be described by assigning to a subset of the velocities (accelerations) each state. Evolutions of such problem are considered in Differential Inclusion. We considered solutions that are absolutely continuous: the evolution is continuous and the set of discontinuities of the derivative of the evolution is a set of measure zero. All such evolutions are path connected when the multivalued map is non-empty, closed, autonomous, and Lipschitz [StWu 91]. Although, Lipschitz condition is fairly weak, the equations of motion are not even continuous at places where a rigid body motion changes the type of contact with another rigid body. A model of impact, however, can be used to predict the initial conditions of the motion after impact. The mechanics of motion in the presence of friction may introduce additional such constraints.

In the region where the set of evolutions are path-connected, we gave conditions for the uniqueness and existence of an initial manifold on the local boundary of the forward and back projections of a regular closed set. This result is a direct application of the Hamilton-Jacobi Theorem to a Hamiltonian system. The boundary of forward and back projections is foliated by the solutions of a Hamiltonian system [BlFi 86, But 82]. They are ordinary differential equations with initial states in the initial manifold. If n is the dimension of the state space, there are $2n$ ordinary differential equations in the system.

The phase portrait of a single-valued non-linear dynamical system is dominated by the behavior of its invariant set: the singularities of the vector field, closed orbits, the stable and unstable manifolds of such singularities, and other non-wandering sets [Smale 67, PalMel 82]. The basin of attraction of a sink singularity is a natural candidate for a Preimage. Such basins of attraction are bounded by the stable and unstable manifolds.

We introduced singular invariant sets for multivalued vector fields that are analogous to the singularities of a single-valued system. We gave conditions when the boundary of the perturbations of the stable and unstable manifolds exist and are unique. They characterize the boundary of the basin of attraction of a sink singular

invariant set for a multivalued vector field.

4.2 Fine Motion Planning

Aside from characterizing control uncertainty, a fine motion planner must also construct a boolean valued function, called termination condition, on the set of possible sensing. The true condition of this function is interpreted as having attained the goal. Interpretations of sensing that are mutually consistent and interpretations that depend on possible initial states of the system were known in this framework. However, the fact that the sensing interpretations also depend on the termination condition is a new observation. We gave an example when embedding this knowledge in the construction of a preimage augments the size of the preimage.

In summary, we gave new results on One-Step Preimage, and a characterization of the control uncertainty for an autonomous non-linear continuous time dynamical system.

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